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THE COALESCENT STRUCTURE OF CONTINUOUS-TIME GALTON-WATSON TREES

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Take a continuous-time Galton-Watson tree. If the system survives until a large time T , then choose k particles uniformly from those alive. What does the ancestral tree drawn out by these k particles look like? Some special cases are known but we give a more complete answer. We concentrate on near-critical cases where the mean number of offspring is $1 + \mu/T$ for some $\mu \in \mathbb{R}$, and show that a scaling limit exists as $T \rightarrow \infty$. Viewed backwards in time, the resulting coalescent process is topologically equivalent to Kingman's coalescent, but the times of coalescence have an interesting and highly non-trivial structure. The randomly fluctuating population size, as opposed to constant size populations where the Kingman coalescent more usually arises, have a pronounced effect on both the results and the method of proof required. We give explicit formulas for the distribution of the coalescent times, as well as a construction of the genealogical tree involving a mixture of independent and identically distributed random variables. In general subcritical and supercritical cases it is not possible to give such explicit formulas, but we highlight the special case of birth-death processes.

1. Introduction. Let L be a random variable taking values in $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. Consider a continuous-time Galton-Watson tree beginning with one initial particle and branching at rate r with offspring distribution L . We will give more details of the model shortly.

Fix a large time T , and condition on the event that at least k particles are alive at time T . Choose k particles uniformly at random (without replacement) from those alive at time T . These particles, and their ancestors, draw out a smaller tree. The general question that we attempt to answer is: what does this tree look like? This is a fundamental question about Galton-Watson trees; several authors have given answers via interesting and contrasting methods for various special cases, usually when $k = 2$. We aim to give a more complete answer with a unified approach that can be adapted to other situations.

Before explaining our most general results we highlight some illuminating examples. Let \mathcal{N}_t be the set of particles that are alive at time t , and write $N_t = \#\mathcal{N}_t$ for the number of particles that are alive at time t . Let $m = \mathbb{E}[L]$ and for each $j \geq 0$ let $p_j = \mathbb{P}(L = j)$. We assume throughout the article, without further mention, that $p_0 + p_1 \neq 1$.

On the event $\{N_T \geq 2\}$, choose a pair of particles $(U_T, V_T) \in \mathcal{N}_T$ uniformly at random (without replacement). Then let $\mathcal{S}(T)$ be the last time at which these uniformly chosen particles shared a common ancestor. If $N_T \leq 1$ then set $\mathcal{S}(T) = 0$.

If $p_0 \in [0, 1)$ and $p_2 = 1 - p_0$, then the model is a birth-death process. In this case we are able to calculate explicitly the distribution of $\mathcal{S}(T)$ conditional on $\{N_T \geq 2\}$. In particular,

- in the supercritical case when $p_2 > p_0$, the law of $\mathcal{S}(T)$ conditional on $\{N_T \geq 2\}$ converges

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as $T \rightarrow \infty$ to a non-trivial distribution with tail satisfying

$$\lim_{T \rightarrow \infty} \mathbb{P}(\mathcal{S}(T) \geq t \mid N_T \geq 2) \sim 2r(m-1)te^{-r(m-1)t} \quad \text{as } t \rightarrow \infty;$$

- in the subcritical case $p_0 > p_2$, the law of $T - \mathcal{S}(T)$ conditional on $\{N_T \geq 2\}$ converges as $T \rightarrow \infty$ to a non-trivial distribution with tail satisfying

$$\lim_{T \rightarrow \infty} \mathbb{P}(T - \mathcal{S}(T) \geq t \mid N_T \geq 2) \sim \left(1 - \frac{2p_2}{3p_0}\right)e^{r(m-1)t} \quad \text{as } t \rightarrow \infty.$$

In the critical case we can work more generally.

- If L has any distribution satisfying $m = \mathbb{E}[L] = 1$ and $\mathbb{E}[L^2] < \infty$, then the law of $\mathcal{S}(T)/T$ conditional on $\{N_T \geq 2\}$ converges as $T \rightarrow \infty$ to a non-trivial distribution on $[0, 1]$ satisfying

$$\lim_{T \rightarrow \infty} \mathbb{P}\left(\frac{\mathcal{S}(T)}{T} \geq t \mid N_T \geq 2\right) = \frac{2(1-t)}{t^2} \left(\log\left(\frac{1}{1-t}\right) - t\right).$$

This last result is known: Durrett [7] gave a power series expansion, and Athreya [4] gave a representation in terms of a geometric number of exponential random variables, both of which agree with our explicit formula. Lambert [16] gave a similar formula for a critical continuous state branching process. Methods involving the excursion representation of continuum random trees were used by Popovic [23], Aldous and Popovic [3], Lambert [17], and Lambert and Popovic [19] to investigate related questions. We give more details in Section 3.2.

Beyond the critical case, we can find a distributional scaling limit when L is *near-critical*. We let the distribution of L depend on T , and write \mathbb{P}_T to signify that the Galton-Watson process now depends on T as a result.

- Suppose that L satisfies $\mathbb{E}_T[L] = 1 + \mu/T + o(1/T)$, $\mathbb{E}_T[L(L-1)] = \beta + o(1)$, and that L^2 is uniformly integrable under \mathbb{P}_T . Then the law of $\mathcal{S}(T)/T$ conditional on $\{N_T \geq 2\}$ converges as $T \rightarrow \infty$ to a non-trivial distribution on $[0, 1]$ satisfying

$$\lim_{T \rightarrow \infty} \mathbb{P}_T\left(\frac{\mathcal{S}(T)}{T} \geq s \mid N_T \geq 2\right) = 2\left(\frac{e^{r\mu(1-s)} - 1}{e^{r\mu(1-s)} - e^{r\mu}}\right) + 2\frac{(e^{r\mu} - 1)(e^{r\mu(1-s)} - 1)}{(e^{r\mu(1-s)} - e^{r\mu})^2} \log\left(\frac{e^{r\mu} - 1}{e^{r\mu(1-s)} - 1}\right).$$

O'Connell [22, Theorem 2.3] gave this result by using a diffusion approximation, relating the near-critical process to a time-changed Yule tree, and adapting the method of Durrett [7] from the critical case. Again, these authors only considered choosing two particles at time T .

All of the above special cases—although they are already interesting in their own right—are just a taster of our general results. The effectiveness and adaptability of our method is demonstrated by the fact that it recovers, in these cases, the results of several separate investigations using different techniques [4, 7, 16, 22]. In our main result (see Theorem 3), we will give a complete description for the genealogical tree of a uniform sample of $k \geq 2$ individuals in near-critical Galton-Watson processes in the large time limit.

We now attempt to describe our general results in a little more detail. For any $k \geq 2$, under a second moment condition on L , we sample k particles without replacement at time T and trace back the tree induced by them and their ancestors. It turns out that if we view this tree backwards in time, then the coalescent process thus obtained is topologically the same as Kingman's coalescent, but has different coalescent rates. We give an explicit joint distribution function for the limiting $k-1$ coalescent times, which are asymptotically independent of the Kingman tree topology: they can be constructed by choosing k independent random variables with a certain distribution and renormalising by the maximum. Equivalently, the coalescent times can be viewed as a mixture of

independent identically distributed random variables. The correlation introduced by this mixture is due to fluctuations in the population size. On the other hand, Kingman's coalescent usually arises from populations where the total number of individuals is kept constant: see for example [27]. One of the biggest hurdles in our proof is to overcome the effect of fluctuations in the population size; we do this using a change of measure $\mathbb{Q}^{k,T}$ under which the coalescent times decorrelate, making calculations easier.

After this article was released, using knowledge of the precise form of our answers, Lambert [18] was able to construct a remarkable method to obtain some of our formulas for *coalescent point processes*. However, [18] assumes binary branching, so whilst it can apply to birth-death processes, it does not cover our main results concerning general near-critical Galton-Watson processes. We discuss this approach further in Section 3.2.

Ren, Song and Sun [25, 24] have also subsequently used a 2-spine approach (involving analogues of our $\mathbb{Q}^{2,T}$) to give elegant probabilistic proofs of Yaglom theorems about the size of the population conditional on survival, both for the discrete time critical Galton-Watson processes [24] and critical superprocesses [25].

In Section 2, we state full details our main results, we present a more intuitive probabilistic construction of the near-critical scaling limit, and we then provide a heuristic explanation and intuitive probabilistic derivation for it. We follow that with discussion of some of the properties of the scaling limit and comparisons to related results in Section 3. In Section 4, we introduce the tools required to prove our results, including a change of measure and a version of Campbell's formula. We then prove our main result for birth-death processes in Section 5, and our main result for near-critical processes in Section 6.

2. Results. We first describe, in more detail, our continuous-time Galton-Watson tree. Under a probability measure \mathbb{P} , we begin with one particle, the root, which we give the label \emptyset . This particle waits an exponential amount of time τ_\emptyset with parameter r , and then instantaneously dies and gives birth to offspring with labels $1, 2, \dots, L_\emptyset$, where L_\emptyset is an independent copy of the random variable L . To be precise, at the time τ_\emptyset the particle \emptyset is no longer alive and its offspring are. These offspring then repeat, independently, this behaviour: each particle u waits an independent exponential amount of time with parameter r before dying and giving birth to offspring $u1, u2, \dots, uL_u$ where L_u is an independent copy of L , and so on. We let $p_j = \mathbb{P}(L = j)$ and $m = \sum_{j=1}^{\infty} jp_j$. Since we will be using more than one probability measure, we will write $\mathbb{P}[\cdot]$ instead of $\mathbb{E}[\cdot]$ for the expectation operator corresponding to \mathbb{P} .

Denote by \mathcal{N}_T the set of all particles alive at time T . For a particle $u \in \mathcal{N}_T$ we let τ_u be the time of its death, and define $\tau_u(T) = \tau_u \wedge T$. If u is an ancestor of v , we write $u \leq v$, and if u is a *strict* ancestor of v (i.e. $u \leq v$ and $u \neq v$) then we write $u < v$. For technical reasons we introduce a graveyard Δ which is not alive (it is not an element of \mathcal{N}_T).

For a particle $u \in \mathcal{N}_t$ and $s \leq t$, let $u(s)$ be the ancestor of u that was alive at time s . For two particles $u, v \in \mathcal{N}_T$, let $\sigma(u, v)$ be the last time at which they shared a common ancestor,

$$\sigma(u, v) = \sup\{t \geq 0 : u(t) = v(t)\}.$$

Now fix $k \in \mathbb{N}$, and at time T , on the event $N_T \geq k$, pick k particles U_T^1, \dots, U_T^k uniformly at random without replacement from \mathcal{N}_T . We let $\mathcal{P}_t^k(T)$ be the partition of $\{1, \dots, k\}$ induced by letting i and j be in the same block if particles U_T^i and U_T^j shared a common ancestor at time t , i.e. if $\sigma(U_T^i, U_T^j) > t$. We order the elements of $\mathcal{P}_t^k(T)$ by their smallest element.

There are two aspects to the information contained in $\mathcal{P}_t^k(T)$. The first is the topological information: given a collection of blocks, which block will split first, and when it does, what will the

new blocks look like? The second is the times at which the splits occur. We will find that in all of our models, the topological information is asymptotically universal and simple to describe, whereas the split times are much more delicate and depend on the parameters of the model. In order to separate out these two aspects, we require some more notation.

Let $\nu_t^k(T)$ be the number of blocks in $\mathcal{P}_t^k(T)$, or equivalently the number of distinct ancestors of U_T^1, \dots, U_T^k that are alive at time t ; that is, $\nu_t^k(T) = \#\{u \in \mathcal{N}_t : u < U_T^i \text{ for some } i \leq k\}$.

For $i = 1, \dots, k-1$ let

$$\mathcal{S}_i^k(T) = \inf\{t \geq 0 : \nu_t^k(T) > i\}.$$

We call $\mathcal{S}_1^k(T) \leq \dots \leq \mathcal{S}_{k-1}^k(T)$ the *split times*. For technical reasons it is often easier to consider the unordered split times; we let $(\tilde{\mathcal{S}}_1^k(T), \dots, \tilde{\mathcal{S}}_{k-1}^k(T))$ be a uniformly random permutation of $(\mathcal{S}_1^k(T), \dots, \mathcal{S}_{k-1}^k(T))$.

For $i = 1, \dots, k-1$ let $P_i^k(T) = \mathcal{P}_{\mathcal{S}_i^k(T)}^k(T)$, and let $\mathcal{H} = \sigma(P_1^k(T), \dots, P_{k-1}^k(T))$, so that \mathcal{H} contains all the topological information about the tree generated by U_T^1, \dots, U_T^k , but almost no information about the split times.

2.1. Birth-death processes. Fix $\alpha \geq 0$ and $\beta > 0$. Suppose that $r = \alpha + \beta$, $p_0 = \alpha/(\alpha + \beta)$ and $p_2 = \beta/(\alpha + \beta)$, with $p_j = 0$ for $j \neq 0, 2$. This is known as a birth-death process with birth rate β and death rate α . Note that since there are only binary splits, if there are at least k particles alive at time T then when we pick k uniformly at random as above there are always exactly $k-1$ distinct split times. Our first theorem gives an explicit distribution for these split times, in the non-critical case and conditional on $\{N_T \geq k\}$.

THEOREM 1. *Suppose that $\alpha \neq \beta$. The unordered split times are independent of \mathcal{H} , and for any $s_1, \dots, s_{k-1} \in (0, T]$ they satisfy*

$$\begin{aligned} & \mathbb{P}(\tilde{\mathcal{S}}_1^k(T) \geq s_1, \dots, \tilde{\mathcal{S}}_{k-1}^k(T) \geq s_{k-1} | N_T \geq k) \\ &= \frac{k(E_0 - \alpha/\beta)^k}{(E_0 - 1)^{k-1}} \left[\frac{1}{(E_0 - \alpha/\beta)} \prod_{i=1}^{k-1} \frac{E_i - 1}{E_i - E_0} + \sum_{j=1}^{k-1} \frac{(E_j - 1)}{(E_j - E_0)^2} \left(\prod_{\substack{i=1 \\ i \neq j}}^{k-1} \frac{E_i - 1}{E_i - E_j} \right) \log \left(\frac{\beta E_0 - \alpha}{\beta E_j - \alpha} \right) \right] \end{aligned}$$

where $E_j = e^{(\beta-\alpha)(T-s_j)}$ for each $j = 1, \dots, k$ and $s_0 = 0$. Furthermore, the partition process $P_0^k(T), P_1^k(T), \dots, P_{k-1}^k(T)$ has the following description:

- if $P_i^k(T)$ contains blocks of sizes a_1, \dots, a_{i+1} , the probability that the next block to split will be block j is $(a_j - 1)/(k - i - 1)$;
- if a block of size a splits, it creates two blocks whose sizes are l and $a - l$ with probability $1/(a - 1)$ for each $l = 1, \dots, a - 1$.

The case of the Yule tree, in which $\beta = 1$ and $\alpha = 0$, gives simpler formulas.

EXAMPLE 1 (Yule tree). Suppose that $\alpha = 0$ and $\beta = 1$. Then for any $s \in (0, T]$,

$$\mathbb{P}(\tilde{\mathcal{S}}_1^2(T) \geq s | N_T \geq 2) = \frac{2(e^{-s} - e^{-T})(e^{-s} - 1 + s)}{(1 - e^{-T})(1 - e^{-s})^2}$$

and for any $s_1, s_2 \in (0, T]$,

$$\begin{aligned} & \mathbb{P}(\tilde{\mathcal{S}}_1^3(T) \geq s_1, \tilde{\mathcal{S}}_2^3(T) \geq s_2 | N_T \geq 3) \\ &= 3(e^{-s_1} - e^{-T})(e^{-s_2} - e^{-T}) \frac{(s_1(1 - e^{-s_2})^2 - s_2(1 - e^{-s_1})^2 + (1 - e^{-s_1})(1 - e^{-s_2})(e^{-s_2} - e^{-s_1}))}{(1 - e^{-T})^2(1 - e^{-s_1})^2(1 - e^{-s_2})^2(e^{-s_2} - e^{-s_1})}. \end{aligned}$$

Returning to general $\alpha \neq \beta$, as mentioned in the introduction, the case $k = 2$ is of particular interest. As there is only one split time when the sample consists of a pair of particles, we can simply write $\mathcal{S}(T) = \mathcal{S}_1^2(T)$. Taking a limit as $T \rightarrow \infty$ simplifies the formula significantly, although we have to consider the supercritical and subcritical cases separately.

EXAMPLE 2 (Supercritical birth-death, $T \rightarrow \infty$). Suppose that $\beta > \alpha$. Then for any $s > 0$,

$$\lim_{T \rightarrow \infty} \mathbb{P}(\mathcal{S}(T) \geq s \mid N_T \geq 2) = \frac{2e^{-(\beta-\alpha)s}}{(1 - e^{-(\beta-\alpha)s})^2} ((\beta - \alpha)s - 1 + e^{-(\beta-\alpha)s}).$$

EXAMPLE 3 (Subcritical birth-death, $T \rightarrow \infty$). Suppose that $\alpha > \beta$. Then for any $s > 0$,

$$\lim_{T \rightarrow \infty} \mathbb{P}(\mathcal{S}(T) \geq T - s \mid N_T \geq 2) = \frac{2\alpha^2}{\beta^2} (e^{(\alpha-\beta)s} - 1) \left(e^{(\alpha-\beta)s} \log \left(1 + \frac{\beta}{\alpha e^{(\alpha-\beta)s} - \beta} \right) - \frac{\beta}{\alpha} \right).$$

To our knowledge all of these results are new. We note (as Durrett also mentioned in [7]) that in the supercritical case, the time $\mathcal{S}(T)$ is likely to be near 0, whereas in the subcritical case, $\mathcal{S}(T)$ is likely to be near T . This much is to be expected, but the detailed behaviour is perhaps more surprising: as mentioned in the introduction, some elementary calculations using the formulas above show that in the supercritical case,

$$\lim_{T \rightarrow \infty} \mathbb{P}(\mathcal{S}(T) \geq s \mid N_T \geq 2) \sim 2(\beta - \alpha)se^{-(\beta-\alpha)s} \quad \text{as } s \rightarrow \infty,$$

whereas in the subcritical case,

$$\lim_{T \rightarrow \infty} \mathbb{P}(T - \mathcal{S}(T) \geq s \mid N_T \geq 2) \sim \left(1 - \frac{2\beta}{3\alpha} \right) e^{-(\alpha-\beta)s} \quad \text{as } s \rightarrow \infty.$$

We can also give analogous results in the critical case $\alpha = \beta$.

THEOREM 2. Suppose that $\alpha = \beta$. The unordered split times are independent of \mathcal{H} , and for any distinct $s_1, \dots, s_{k-1} \in (0, T]$ with $s_i \neq s_j$ for any $i \neq j$,

$$\begin{aligned} & \mathbb{P}(\tilde{\mathcal{S}}_1^k(T)/T \geq s_1, \dots, \tilde{\mathcal{S}}_{k-1}^k(T)/T \geq s_{k-1} \mid N_T \geq k) \\ &= k \left(1 + \frac{1}{\beta T} \right)^k \left[\frac{1}{1 + 1/T} \prod_{i=1}^{k-1} \left(1 - \frac{1}{s_i} \right) + \sum_{j=1}^{k-1} \frac{1 - s_j}{s_j^2} \left(\prod_{\substack{i=1 \\ i \neq j}}^{k-1} \frac{1 - s_i}{s_j - s_i} \right) \log \left(\frac{1 + 1/T}{1 - s_j + 1/T} \right) \right]. \end{aligned}$$

Furthermore, the partition process $P_0^k(T), P_1^k(T), \dots, P_{k-1}^k(T)$ has the following description:

- if $P_i^k(T)$ contains blocks of sizes a_1, \dots, a_{i+1} , the probability that the next block to split will be block j is $(a_j - 1)/(k - i - 1)$;
- if a block of size a splits, it creates two blocks whose sizes are l and $a - l$ with probability $1/(a - 1)$ for each $l = 1, \dots, a - 1$.

EXAMPLE 4. Suppose that $\alpha = \beta$. Then for any $s > 0$

$$\mathbb{P}(\tilde{\mathcal{S}}_1^2(T)/T \geq s \mid N_T \geq 2) = 2 \left(1 + \frac{1}{\beta T} \right)^2 \left(\frac{1 - s}{s^2} \right) \left(\log \left(\frac{1 + 1/T}{1 - s + 1/T} \right) - \frac{s}{1 + 1/T} \right)$$

and for any $s_1, s_2 > 0$,

$$\begin{aligned} & \mathbb{P}(\tilde{\mathcal{S}}_1^3(T)/T \geq s_1, \tilde{\mathcal{S}}_2^3(T)/T \geq s_2 \mid N_T \geq 3) \\ &= \frac{3(1 + \frac{1}{\beta T})^3(1 - s_1)(1 - s_2)}{s_1^2 s_2^2 (s_2 - s_1)} \left[s_2^2 \log \left(\frac{1 - s_1 + \frac{1}{T}}{1 + \frac{1}{T}} \right) - s_1^2 \log \left(\frac{1 - s_2 + \frac{1}{T}}{1 + \frac{1}{T}} \right) + \frac{s_1 s_2 (s_2 - s_1)}{1 + \frac{1}{T}} \right]. \end{aligned}$$

We can easily let $T \rightarrow \infty$ in these formulas, but in cases near criticality, if we are willing to take a scaling limit as $T \rightarrow \infty$ then we can work much more generally.

2.2. Near-critical processes: a scaling limit. We no longer restrict to birth-death processes; the birth distribution L may take any non-negative integer value. In order to get a scaling limit, we take Galton-Watson processes that are *near-critical* in that the mean number of offspring is approximately $1 + \mu/T$ when T is large. Henceforth, we will assume the following:

ASSUMPTION 1. For some $\mu \in \mathbb{R}$ and $\sigma > 0$, suppose that for each $T > 0$, the offspring distribution L satisfies:

- $\mathbb{P}_T[L] = 1 + \mu/T + o(1/T)$
- $\mathbb{P}_T[L(L-1)] = \sigma^2 + o(1)$
- L^2 is uniformly integrable under \mathbb{P}_T (i.e. $\forall \varepsilon > 0, \exists K$ such that $\mathbb{P}_T[L^2 \mathbb{1}_{\{L > K\}}] < \varepsilon \forall T$)

where, for \mathbb{R} -valued functions f and g , $f(x) = o(g(x))$ means that $f(x)/g(x) \rightarrow 0$ as $x \rightarrow \infty$.

Conditional on the population surviving to a large time T , we sample k particles uniformly without replacement and wish to understand their genealogical tree. Our near critical Galton-Watson process conditioned to survive for a large time produces a large population that fluctuates naturally over time. In other branching models with *constant* population size, it has been shown that the genealogical tree emerging in the large population limit is Kingman's coalescent; see [27]. We find something significantly different and more complex.

THEOREM 3 (Near-critical scaling limit). *Suppose that Assumption 1 holds. Then the split times are asymptotically independent of \mathcal{H} , and if $\mu \neq 0$, then for any $s_1, \dots, s_{k-1} \in (0, 1)$ with $s_i \neq s_j$ for any $i \neq j$,*

$$\begin{aligned} \lim_{T \rightarrow \infty} \mathbb{P}_T(\tilde{\mathcal{S}}_1^k(T)/T \geq s_1, \dots, \tilde{\mathcal{S}}_{k-1}^k(T)/T \geq s_{k-1} \mid N_T \geq k) \\ = k \prod_{i=1}^{k-1} \frac{E_i}{E_i - E_0} + k \sum_{j=1}^{k-1} \frac{E_0 E_j}{(E_j - E_0)^2} \left(\prod_{\substack{i=1 \\ i \neq j}}^{k-1} \frac{E_i}{E_i - E_j} \right) \log \frac{E_0}{E_j} \end{aligned}$$

where $E_j = e^{r\mu(1-s_j)} - 1$ for each $j = 0, \dots, k-1$ and $s_0 = 0$. If $\mu = 0$, then instead

$$\begin{aligned} \lim_{T \rightarrow \infty} \mathbb{P}(\tilde{\mathcal{S}}_1^k(T)/T \geq s_1, \dots, \tilde{\mathcal{S}}_{k-1}^k(T)/T \geq s_{k-1} \mid N_T \geq k) \\ = k \prod_{i=1}^{k-1} \frac{s_i - 1}{s_i} - k \sum_{j=1}^{k-1} \frac{1 - s_j}{s_j^2} \left(\prod_{\substack{i=1 \\ i \neq j}}^{k-1} \frac{1 - s_i}{s_j - s_i} \right) \log(1 - s_j). \end{aligned}$$

Furthermore, the partition process $P_0^k(T), P_1^k(T), \dots, P_{k-1}^k(T)$ has the following description:

- if $P_i^k(T)$ contains blocks of sizes a_1, \dots, a_{i+1} , the probability that the next block to split will be block j converges as $T \rightarrow \infty$ to $(a_j - 1)/(k - i - 1)$;
- if a block of size a splits, with probability tending to 1 it creates two blocks whose sizes are l and $a - l$ with probability converging to $1/(a - 1)$ for each $l = 1, \dots, a - 1$.

In Theorems 1 and 2 we saw that the split times were independent of \mathcal{H} . This cannot be the case in Theorem 3, since two or more split times may be equal with positive probability, an event

which is captured by both the split times and the topological information \mathcal{H} . However we do see that the split times are *asymptotically* independent, in that $\mathbb{P}_T(A \cap B) \rightarrow \mathbb{P}_T(A)\mathbb{P}_T(B)$ for any $A \in \sigma(\mathcal{S}_1^k(T), \dots, \mathcal{S}_{k-1}^k(T))$ and $B \in \mathcal{H}$, which is the best that we can hope for.

We note here that the topology of the (limiting) tree described forwards in time in Theorem 3 is the same as that described backwards in time by Kingman's coalescent; but the times of splits (or times of mergers, in the coalescent picture) are drastically different.

EXAMPLE 5 (Critical processes). Suppose that $\mathbb{P}[L] = 1$ and $\mathbb{P}[L^2] < \infty$. Then for any $s \in (0, 1)$,

$$(1) \quad \lim_{T \rightarrow \infty} \mathbb{P}(\mathcal{S}(T)/T \geq s \mid N_T \geq 2) = \frac{2(1-s)}{s^2} \left(\log \left(\frac{1}{1-s} \right) - s \right).$$

EXAMPLE 6 (Near-critical scaling limit, $k = 2$). Suppose that the conditions of Theorem 3 hold with $\mu \neq 0$. Then for any $s \in (0, 1)$, as $T \rightarrow \infty$,

$$\mathbb{P}_T(\mathcal{S}(T)/T \geq s \mid N_T \geq 2) \rightarrow 2 \left(\frac{e^{r\mu(1-s)} - 1}{e^{r\mu(1-s)} - e^{r\mu}} \right) + 2 \frac{(e^{r\mu} - 1)(e^{r\mu(1-s)} - 1)}{(e^{r\mu(1-s)} - e^{r\mu})^2} \log \left(\frac{e^{r\mu} - 1}{e^{r\mu(1-s)} - 1} \right).$$

Both these examples are known [22], but to our knowledge the general formula is not. We give more information on related results in Section 3.1.

2.3. *Construction of the near-critical scaling limit.* In this section we investigate further the scaling limit observed in Theorem 3. Our aim is to give a more intuitive probabilistic understanding of the scaling limit, rather than the explicit formulas seen in Theorems 1 to 3. We continue to work under Assumption 1, given in Section 2.2.

Theorem 3 says that the rescaled unordered split times, conditional on at least k particles being alive at time T , converge jointly in distribution to an explicit limit,

$$\left(\frac{\tilde{\mathcal{S}}_1^k(T)}{T}, \dots, \frac{\tilde{\mathcal{S}}_{k-1}^k(T)}{T} \right) \xrightarrow{(d)} (\tilde{\mathcal{S}}_1^k, \dots, \tilde{\mathcal{S}}_{k-1}^k).$$

We aim to shed some more light on this limit. First we note that, although the split times (for fixed T) do not usually have a joint density—with positive probability one split time may equal another—their scaling limit *does* have a density. Indeed, from the proof of Theorem 3 (or by checking directly) we see that this density satisfies (with $s_0 = 0$)

$$f_k(s_1, \dots, s_{k-1}) = \begin{cases} k(r\mu)^{k-1}(1 - e^{-r\mu}) \int_0^\infty \theta^{k-1} \prod_{i=0}^{k-1} \frac{e^{r\mu(1-s_i)}}{(1 + \theta(e^{r\mu(1-s_i)} - 1))^2} d\theta & \text{if } \mu > 0 \\ k \int_0^\infty \theta^{k-1} \prod_{i=0}^{k-1} \frac{1}{(1 + \theta(1 - s_i))^2} d\theta & \text{if } \mu = 0 \\ k(-1)^k(r\mu)^{k-1}(1 - e^{-r\mu}) \int_0^\infty \theta^{k-1} \prod_{i=0}^{k-1} \frac{e^{r\mu(1-s_i)}}{(1 - \theta(e^{r\mu(1-s_i)} - 1))^2} d\theta & \text{if } \mu < 0. \end{cases}$$

The following two theorems give constructions of the scaling limits of the tree in the spirit of Aldous' construction of Kingman's coalescent [2, Section 4.2]. In particular it gives a method for consistently constructing the times $(\tilde{\mathcal{S}}_1^k, \dots, \tilde{\mathcal{S}}_{k-1}^k)$.

THEOREM 4 (A construction for critical genealogies). *Suppose that $\mu = 0$. Let X_1, X_2, \dots be a sequence of independent and identically distributed random variables on $(0, \infty)$ with density $(1+x)^{-2}$. Let $M_k = \max_{i \leq k} X_i$, and choose I such that $X_I = M_k$. For $i \leq k$ define $T_i = 1 - X_i/M_k$. Then $(T_1, \dots, T_{I-1}, T_{I+1}, \dots, T_k)$ is equal in distribution to $(\tilde{S}_1^k, \dots, \tilde{S}_{k-1}^k)$.*

Moreover, the ancestral tree drawn out by the k uniformly chosen particles has the following description: let U_1, U_2, \dots be independent uniform random variables on $[0, 1]$. Within the unit square, for each $1 \leq i \leq k$, draw a vertical line from $(U_i, 0)$ to $(U_i, 1 - T_i)$. These lines represent the branches of our tree. Now, for each $1 \leq i \leq k-1$, draw a horizontal line starting from (U_i, T_i) towards (U_I, T_i) but stopping as soon as it hits another vertical line (see Figure 1).

This result, in particular, clarifies the consistency of the split times. Of course, if we choose $k+1$ particles uniformly without replacement at time T , and then forget one of them, the result should be consistent with choosing k particles originally. This is not immediately obvious from the distribution in Theorem 3, but it follows easily from the construction in Theorem 4. In fact, the particular choice of U_1, U_2, \dots is not so important above; they simply provide a convenient way to consistently construct random permutations of $\{1, \dots, k\}$ for every $k \in \mathbb{N}$.

THEOREM 5 (A construction for near-critical genealogies). *Suppose that $\mu \neq 0$. Let X_1, X_2, \dots be a sequence of independent and identically distributed random variables on $(0, \infty)$ with density $(1+x)^{-2}$. Let $M_k = \max_{i \leq k} X_i$, and choose I such that $X_I = M_k$. For $i \leq k$ define*

$$T_i = 1 - \frac{1}{r\mu} \log \left(1 + (e^{r\mu} - 1) \frac{X_i}{M_k} \right).$$

Then $(T_1, \dots, T_{I-1}, T_{I+1}, \dots, T_k)$ is equal in distribution to $(\tilde{S}_1^k, \dots, \tilde{S}_{k-1}^k)$. Moreover, the ancestral tree drawn out by the k particles has the same construction as in Theorem 4.

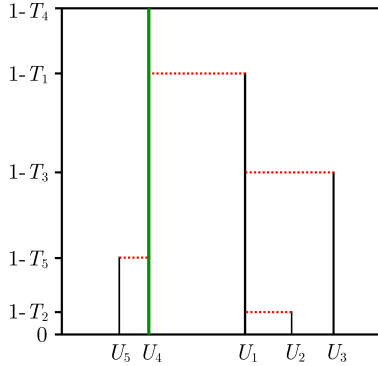


Fig 1: A representation of the rescaled tree drawn out by 5 particles chosen uniformly at random from those alive at a large time. Here $I = 4$.

2.4. Heuristic explanation of our results. In this section, we aim to give a quick intuitive probabilistic derivation of Theorem 4. For this we will need to use a certain very natural probability measure, $\mathbb{Q}^{k,T}$. Whilst $\mathbb{Q}^{k,T}$ will not be defined until Section 4 (see (6)), and is fundamental to our approach, for now it will be sufficient to know only a few of its basic properties. The probability measure $\mathbb{Q}^{k,T}$ will describe the behaviour of k distinguished *spine* particles along which standard Galton-Watson processes are immigrated. Under $\mathbb{Q}^{k,T}$, these k spines will have the property of looking like a uniform choice without replacement from the N_T particles alive at time T . For this

heuristic we will use this measure $\mathbb{Q}^{k,T}$, together with the classical theorems of Kolmogorov [15] about the asymptotics of the survival probability, and Yaglom [28] about the distribution of the scaled population size conditioned to survive.

Let E_k be any event concerning the tree drawn out by the k uniformly sampled particles (we will only consider these conditionally on $N_T \geq k$ so that they always exist). It will be easy to show, using the definition of our change of measure $\mathbb{Q}^{k,T}$, that

$$(2) \quad \mathbb{P}(E_k | N_T \geq k) = \mathbb{Q}^{k,T} \left[\frac{\mathbb{1}_{E_k^\xi}}{N_T(N_T - 1) \dots (N_T - k + 1)} \right] \frac{\mathbb{P}[N_T(N_T - 1) \dots (N_T - k + 1)]}{\mathbb{P}(N_T \geq k)}$$

where E_k^ξ is the event corresponding to E_k , but for the k spines under $\mathbb{Q}^{k,T}$, rather than the k uniformly chosen particles under \mathbb{P} .

Now, the second fraction above can be approximated using Yaglom's theorem: as $T \rightarrow \infty$,

$$(3) \quad \frac{\mathbb{P}[N_T(N_T - 1) \dots (N_T - k + 1)]}{\mathbb{P}(N_T \geq k)} = \mathbb{P}[N_T(N_T - 1) \dots (N_T - k + 1) | N_T \geq k] \\ \sim T^k \mathbb{P}[(N_T/T)^k | N_T > 0] \sim T^k \mathbb{P}[\mathcal{E}^k]$$

where \mathcal{E} is an exponential random variable with parameter $2/\sigma^2$. Therefore, in order to describe the distribution of the tree drawn out by the k uniformly sampled particles under \mathbb{P} when T is large, it suffices to understand the joint distribution of the tree drawn out by the k spines together with N_T under $\mathbb{Q}^{k,T}$ when T is large.

Write $\tau_i = \tilde{S}_i^k(T)/T$ for the scaled split times of the k uniformly sampled particles, and τ_i^ξ for the scaled split times of the k spine (unordered, in the sense that they are a random permutation of the ordered split times). In Proposition 27, Lemma 28, and also the discussion in Section 4.4, we will see that in the limit as $T \rightarrow \infty$ under $\mathbb{Q}^{k,T}$, the times $(\tau_1^\xi, \dots, \tau_{k-1}^\xi)$ are uniform random variables on $[0, 1]$, and the topology of the underlying tree is equivalent to the topology of Kingman's coalescent restricted to k blocks. Here is a way of constructing such a tree, again in the spirit of Aldous [2, Section 4.2] and similar to Figure 1. Let U_0, \dots, U_{k-1} and V_1, \dots, V_{k-1} be independent uniform random variables on $[0, 1]$. Also let $V_0 = 1$. Within the unit square, for each $0 \leq i \leq k-1$, draw a line from $(U_i, 0)$ to (U_i, V_i) . These lines represent the branches of our tree. Now, for each $1 \leq i \leq k-1$, draw a horizontal line starting from (U_i, V_i) towards (U_0, V_i) but stopping as soon as it hits another (vertical) line. This is our description of the tree drawn out by the spines under $\mathbb{Q}^{k,T}$ as $T \rightarrow \infty$. (Note, as earlier, the particular choice of the U_i is merely a convenient way to give a random permutation of the vertical lines; the scale on the horizontal axis has no meaning here yet.)

Now we explain how to observe the joint distribution of this tree and the total population size, given the description above. Under $\mathbb{Q}^{k,T}$, each spine—that is, each vertical line in our picture—behaves in the same way, giving birth to ordinary particles at a constant rate (independent of the number of marks following the spine); this can be seen from Lemma 9. Thus the contribution to the total population of a vertical line of length v in our picture is simply the contribution to the total population of a single spine that lived for time vT . It is immediate from the definition of $\mathbb{Q}^{1,vT}$ that a single spine results in a size-biasing of the total population size; by Yaglom's theorem, under \mathbb{P} , the total population size after time vT is approximately vT times an independent exponential random variable of parameter $2/\sigma^2$, and therefore under $\mathbb{Q}^{1,vT}$ the total population size is approximately vT times an independent Gamma random variable of parameters $(2, 2/\sigma^2)$.

Thus the total population size N_T under $\mathbb{Q}^{k,T}$ satisfies

$$\frac{N_T}{T} \xrightarrow{(d)} \sum_{i=0}^{k-1} V_i \Gamma_i$$

where the branch lengths V_1, \dots, V_{k-1} are independent $U[0, 1]$ random variables, $V_0 = 1$, and $\Gamma_0, \dots, \Gamma_{k-1}$ are independent identically distributed $\Gamma(2, 2/\sigma^2)$ random variables. Now, a uniform random variable multiplied by an independent $\Gamma(2, 2/\sigma^2)$ random variable is exponentially distributed with parameter $2/\sigma^2$; that is, $\mathcal{E}_i := V_i \Gamma_i \sim \text{Exp}(2/\sigma^2)$ for $i = 1, \dots, k-1$. As $V_0 = 1$, $V_0 \Gamma_0$ is distributed as the sum of two independent exponential random variables, say \mathcal{E}_0 and \mathcal{E}'_0 , each with parameter $2/\sigma^2$. Thus, the total population size under $\mathbb{Q}^{k,T}$ is approximately T times a sum of $k+1$ independent $\text{Exp}(2/\sigma^2)$ random variables, or in other words, T times a $\Gamma(k+1, 2/\sigma^2)$ random variable. We can now re-draw our probabilistic representation of the rescaled tree under $\mathbb{Q}^{k,T}$ for large T to jointly include the *topology*, *split times*, and *sub-populations*; see Figure 2. (Note, the new horizontal scale corresponds to sub-population sizes; here, we only used the earlier U values to give the random ordering of the vertical lines.)

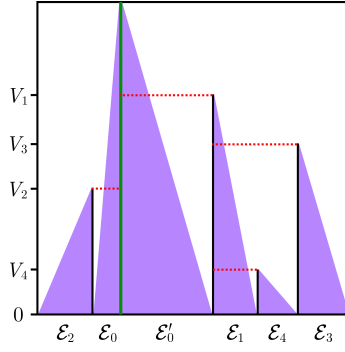


Fig 2: A probabilistic representation of the rescaled tree under $\mathbb{Q}^{5,T}$ for large T . Each triangle represents the contribution towards the total population from particles that branched off the adjacent spine. The random variables drawn on the horizontal axis can be interpreted as population size.

To complete the explanation of our results, continuing from (2) and (3), we now see that

$$\begin{aligned} \mathbb{P}(\tau_1 \in dt_1, \dots, \tau_{k-1} \in dt_{k-1} | N_T \geq k) &\sim \mathbb{Q}^{k,T} \left[\frac{\mathbb{I}_{\{\tau_1^\xi \in dt_1, \dots, \tau_{k-1}^\xi \in dt_{k-1}\}}}{N_T(N_T - 1) \dots (N_T - k + 1)} \right] T^k \mathbb{P}[\mathcal{E}^k] \\ &\sim \mathbb{P} \left[\frac{\mathbb{I}_{\{1 - V_1 \in dt_1, \dots, 1 - V_{k-1} \in dt_{k-1}\}}}{T^k (\sum_{i=0}^{k-1} V_i \Gamma_i)^k} \right] T^k \mathbb{P}[\mathcal{E}^k] \\ &= \mathbb{P} \left[\frac{1}{(\sum_{i=0}^{k-1} (1 - t_i) \Gamma_i)^k} \right] \mathbb{P}[\mathcal{E}^k] dt_1 \dots dt_{k-1}. \end{aligned}$$

We note the fact that, for any $\alpha > 0$, $\frac{1}{\alpha^k} = \frac{1}{(k-1)!} \int_0^\infty z^{k-1} e^{-\alpha z} dz$. Thus, we find

$$\begin{aligned} \mathbb{P}(\tau_1 \in dt_1, \dots, \tau_{k-1} \in dt_{k-1} | N_T \geq k) &\sim \mathbb{P} \left[\frac{1}{(k-1)!} \int_0^\infty z^{k-1} e^{-z \sum_{i=0}^{k-1} (1-t_i) \Gamma_i} dz \right] \mathbb{P}[\mathcal{E}^k] dt_1 \dots dt_{k-1} \\ &= \frac{1}{(k-1)!} \int_0^\infty z^{k-1} \prod_{i=0}^{k-1} \frac{1}{(1 + \frac{\sigma^2}{2} (1-t_i) z)^2} dz k! \left(\frac{\sigma^2}{2} \right)^k dt_1 \dots dt_{k-1} \\ &= k \int_0^\infty z^{k-1} \prod_{i=0}^{k-1} \frac{1}{(1 + (1-t_i) z)^2} dz dt_1 \dots dt_{k-1}. \end{aligned}$$

Indeed, this is the joint density of the coalescent times in the critical case as given in Section 2.3,

and consistent with the construction in Theorem 4. Further, integrating gives the joint distribution function in Theorem 2.

Note that in near-critical cases a similar picture will hold, although the distribution of the rescaled spine split times will not be uniform and will have a density that is proportional to $e^{r\mu(1-s)}$ for $s \in [0, 1]$. See Section 6 for more details.

3. Further discussion of the results. In this section we try to understand our scaling limit further, compare it to known results, and explore other ways of obtaining similar representations. For brevity, we will not worry too much about technical details. We will return to full rigour in Sections 4, 5 and 6, in order to prove our main results.

3.1. Comparison to known formulas. As mentioned in the introduction, the critical case $\mu = 0$ has been investigated by other authors. For $k = 2$, Athreya [4] gave an implicit description of the distributional limit of $\mathcal{S}(T)/T$. In fact, he worked with discrete-time Galton-Watson processes, but this makes no difference in the limit, and we will continue to use our continuous-time terminology and notation for ease of comparison. By considering the numbers of descendants at time T of particles alive at an earlier time sT , Athreya showed that

$$\lim_{T \rightarrow \infty} \mathbb{P}(\mathcal{S}(T)/T < s \mid N_T \geq 2) = 1 - E[\phi(G_s)]$$

where G_s satisfies $P(G_s = j) = (1 - s)s^{j-1}$ for $j \geq 1$, and

$$\phi(j) = E \left[\frac{\sum_{i=1}^j \eta_i^2}{(\sum_{i=1}^j \eta_i)^2} \right]$$

where η_1, η_2, \dots are independent exponential random variables of parameter 1.

One can show using properties of the exponential distribution that $\phi(j) = 2/(j+1)$, and a simple computation shows that this description of the scaling limit agrees with our own formula (1). We omit this calculation here, but it is available in full as Lemma 6 of [11].

Durrett [7] also gave a description for the limit of $\mathcal{S}(T)/T$ in the critical case, showing that

$$\lim_{T \rightarrow \infty} \mathbb{P}(\mathcal{S}(T)/T > s \mid N_T \geq 2) = (1 - s) \left(1 + 2 \sum_{j=1}^{\infty} \frac{s^j}{j+2} \right).$$

It is easy to expand our formula (1) as a power series and check that it agrees with the above. Durrett, in fact, went on to give power series expressions when $k = 3$ for the distributions of \mathcal{S}_1^3 and \mathcal{S}_2^3 . He further stated that it was “theoretically” possible to calculate distributions of split times for $k > 3$, and also mentioned that he could derive a joint distribution for \mathcal{S}_1^3 and \mathcal{S}_2^3 , again in power series form, but that “we would probably not obtain a useful formula”. This makes clear the advantage of our method, which gives explicit formulas for the joint distribution for each k without going through an iterative procedure.

O’Connell [22] gave exactly the formula in our Example 6, the near-critical scaling limit in the case $k = 2$. He also provided a very interesting application to a biologically motivated problem: how long ago did the most recent common ancestor of all humans live?

In subcritical and supercritical cases, it is impossible to give such explicit results in generality as the genealogical structure of the tree depends on the detail of the offspring distribution. However one can characterize the distribution of the split times using integral formulas involving the generating function of the offspring distribution. Lambert [16] (in discrete time) and Le [21] (in continuous time) did this in the case $k = 2$ for quite general Galton-Watson processes. Le also investigated

the case $k \geq 3$, but gave only an implicit representation for the joint distribution of the split times. More recently Grosjean and Huillet [10] and Johnston [14] gave detailed answers for general k .

Donnelly and Kurtz [6, Theorem 5.1] showed that the genealogy of the Feller diffusion is a time-change of Kingman's coalescent, in which the rate at which two lineages merge is inversely proportional to the population size. The Feller diffusion started from x is itself the scaling limit of a critical Galton-Watson process started with a population of size $\lfloor Nx \rfloor$, so taking a limit as $x \downarrow 0$ one might expect to be able to recover our results. However, finding the marginal distribution of the coalescent times—that is, *not* conditional on the population size—is highly non-trivial, as the two quantities are so closely connected; this can be seen in (2), for example. We manage to overcome this serious difficulty by decoupling the dependence between the population size and the split times via the measure $\mathbb{Q}^{k,T}$, which adjusts for the varying population size whilst simultaneously ensuring the k spines form a uniform sample without replacement from population at time T .

Besides being more difficult, the question of understanding the distribution of the coalescent tree drawn out by a sample from a large population, without knowing the population size, appears to be more natural from the point of view of biological applications.

3.2. Contour processes and the continuum random tree. It is known that a critical Galton-Watson tree conditioned to survive until time T converges, as $T \rightarrow \infty$ (in a suitable topology), to a *continuum random tree*. There is a vast literature, beginning with Aldous [1], on continuum random trees. For our discussion we can think of drawing our tree, conditioned to survive to time T and renormalised by T , and tracing a contour around it starting from the root and proceeding in a depth-first manner from left to right. The height of that contour process converges as $T \rightarrow \infty$ to a Brownian excursion $(B_t)_{t \in [0, \nu]}$ conditioned to reach height 1. It is easy to see that two points $u, v \in [0, \nu]$ correspond to the same “vertex” in the limiting tree if they are at the same height and the excursion between u and v is always above B_u . The total population of the tree at time sT corresponds to the local time of the Brownian excursion at level s . Choosing two particles at time T means picking two points on the excursion at height 1 according to the local time measure; and the two particles have a common ancestor at time t if the two points chosen are in the same sub-excursion above height t .

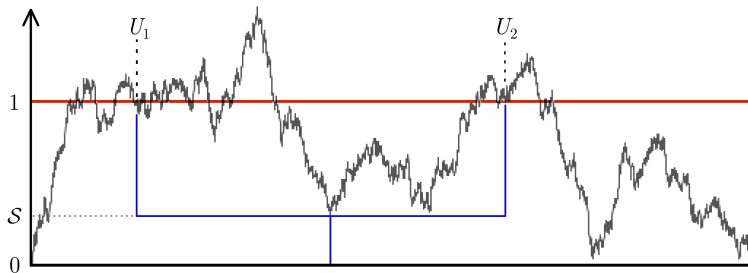


Fig 3: A Brownian excursion conditioned to reach height 1. Two points U_1 and U_2 are chosen uniformly according to local time at height 1, and the induced tree is drawn below the excursion.

In order to calculate the probability of this last event, we (obviously) need to know a little about Brownian excursions. Excursions, indexed by local time, occur according to a Poisson point process with intensity Lebesgue $\times n$ for some excursion measure n . This measure n satisfies $n(\sup_t f(t) > a) = \frac{1}{2a}$; and the local time at 0 when the Brownian motion first hits $-\delta$ is exponentially distributed with parameter $\frac{1}{2\delta}$. See for example [26].

Take a Brownian excursion conditioned to reach height 1, and choose two points U_1 and U_2 at height 1 uniformly according to local time measure. Let L_1 be the total local time at level 1, and L_U

be the total local time between U_1 and U_2 . The event that U_1 and U_2 are in the same sub-excursion above height s is exactly the event that there is no excursion from level 1 between U_1 and U_2 that goes below level s (and stays above level 0); by the facts about Brownian excursions above, given L_U , the number of such excursions is a Poisson random variable with parameter $L_U(\frac{1}{2(1-s)} - \frac{1}{2})$. Thus the probability that U_1 and U_2 are in the same sub-excursion above height s is

$$\int_0^\infty \mathbb{P}(L_1 \in dx) \int_0^x \mathbb{P}(L_U \in dy \mid L_1 = x) e^{-y(\frac{1}{2(1-s)} - \frac{1}{2})}.$$

The local time L_1 is exponential of parameter $1/2$, and the density of the distance between two uniform random variables on $(0, x)$ is $2(x - y)/x^2$. Thus the above equals

$$\int_0^\infty \frac{1}{2} e^{-x/2} \int_0^x \frac{2(x - y)}{x^2} e^{-y(\frac{1}{2(1-s)} - \frac{1}{2})} dy dx.$$

Making the substitution $z = y/x$ and swapping integrals, it is easy to integrate directly to obtain the distribution of the limiting split time \mathcal{S} and check that it agrees with (1).

Applying this excursion machinery works well in this simple case. However it becomes much more difficult to generalise these techniques to obtain the joint distribution of the split times for three particles; let alone the general formula for k particles that appeared in Theorem 3.

Popovic [23] used the following observation. Condition on the event that there are exactly k particles alive at time T_k , so that the k particles we choose comprise the whole population, then rescale by T_k and let $k \rightarrow \infty$. If $T_k/k \rightarrow t$, then the contour process converges to a Brownian excursion conditioned to have local time 1 at level t ; and the split times are then governed by the entire collection of excursions below level t . These excursions form a Poisson point process with an explicit intensity measure. This allowed Popovic to give some very interesting results about critical processes, and similar techniques were built upon in various ways by her and other authors [3, 9, 17, 19]. Although these are related to our investigation, they often look at the entire population alive at time T , rather than sampling a fixed number of individuals, which results in a different scaling regime. Biological motivation for why we might like to sample a fixed number of individuals from a fluctuating population—that is, our regime—can be found in [22].

After this article was released, Lambert [18] constructed a remarkable method for obtaining some of our formulas from contour processes. Given a branching process whose population at time T is geometrically distributed (for example a birth-death process), the work in [20] allows one to sample each particle at time T independently with some fixed probability $y \in (0, 1)$ and reconstruct the genealogical tree of the sampled particles. By taking y to be a realisation of a carefully chosen improper random variable Y , and conditioning the resulting number of particles sampled to be exactly k , in [18] Lambert produces our Proposition 19. However, constructing the correct (improper) distribution for Y would have been extremely difficult without prior knowledge of the answers provided by our results.

Lambert's results in [18] are for a large class of processes known as *coalescent point processes*. However, coalescent point processes necessarily have geometrically distributed population sizes. As Lambert says in [18], “we consider here possibly non-Markovian and time-inhomogeneous branching processes, but always binary.” For Galton-Watson processes, this means only our birth-death process results are in common with Lambert's coalescent point process results in [18]. In a more recent private communication, Lambert has told us that he can carry out his construction even in non-binary cases, and that his results hold beyond geometrically distributed population sizes.

Another advantage of our approach is that it does not require a Markovian contour process, and could be generalised, for example, to Galton-Watson processes with infinite variance, or spatial branching processes. We plan to carry out these generalisations in future work.

3.3. *Reduced trees.* For a moment forget about the scaling limit, and consider a birth-death process (that is, fix $\alpha \geq 0$ and $\beta > 0$, and suppose that $r = \alpha + \beta$, $p_0 = \alpha/(\alpha + \beta)$ and $p_2 = \beta/(\alpha + \beta)$, with $p_j = 0$ for $j \neq 0, 2$). Colour any particle that has a descendant alive at time T purple.

The purple tree, often called the *reduced tree* in the literature, was first introduced by Fleischmann and Siegmund-Schultze [8] and is used in several of the references given in Section 3.1, in particular O’Connell [22]. Harris, Hesse and Kyprianou [12] used a similar construction for supercritical branching processes.

Now suppose that, rather than running the birth-death process until time T and then colouring the particles, we want to construct the coloured picture dynamically as the process evolves. If we start with one particle and condition on the process surviving until time T , then the first particle is certainly purple, since at least one of its descendants must survive.

Let $p_t = \mathbb{P}(N_t = 0)$. Using generating functions one can show that

$$p_t = \frac{\alpha e^{(\beta - \alpha)t} - \alpha}{\beta e^{(\beta - \alpha)t} - \alpha}, \quad 1 - p_t = \frac{(\beta - \alpha)e^{(\beta - \alpha)t}}{\beta e^{(\beta - \alpha)t} - \alpha};$$

see Section 5.1 for details. If a purple particle branches at time s , then it must have either one or two purple children. The probability that they are both purple is $(1 - p_{T-s})^2/(1 - p_{T-s}^2)$, corresponding to the probability that *both* descendancies survive given that at least one does. Similarly one can calculate the probability that exactly one is purple; that purple particles branch at rate $\beta(1 + p_{T-s})$ at time s ; and that other particles branch at rate βp_{T-s} at time s . In particular purple particles give birth to new purple particles at rate

$$\beta(1 + p_{T-s}) \cdot \frac{(1 - p_{T-s})^2}{1 - p_{T-s}^2} = \beta(1 - p_{T-s}).$$

Similar calculations can also be done generally, rather than just for birth-death processes. See [11, Section 3.3] and [12] for more details.

Of course, to understand the coalescent structure of the tree drawn out by particles chosen at time T , we can ignore the red particles. Let us now return to a near-critical scaling limit by assuming that $\beta = \alpha + \gamma/T$ for some $\gamma \neq 0$. Scaling time $[0, T]$ onto $[0, 1]$ and letting $T \rightarrow \infty$, at time $s \in (0, 1)$ one can check that the purple tree undergoes binary branching at rate

$$(4) \quad \lim_{T \rightarrow \infty} T \beta(1 - p_{T(1-s)}) = \frac{\gamma e^{\gamma(1-s)}}{e^{\gamma(1-s)} - 1}.$$

Thus we see that the purple tree in the near-critical scaling limit is the same as a Yule tree (binary branching at rate 1) observed under the time change

$$t \mapsto \int_0^t \frac{\gamma e^{\gamma(1-s)}}{e^{\gamma(1-s)} - 1} ds = \log \left(\frac{e^\gamma - 1}{e^{\gamma(1-t)} - 1} \right).$$

Following the same route in the critical case $\alpha = \beta$ gives that the rescaled purple tree branches at rate $(1 - s)^{-1}$, corresponding to a Yule tree under the time change $t \mapsto -\log(1 - t)$.

These calculations help to explain the similarities between our formulas in the near-critical scaling limit (Theorem 3) and in the birth-death process (Theorem 1). In particular, for the coalescence behaviour, only the purple tree matters. In the large time T limit, only binary branching occurs in the purple tree, since the chance of any purple particle having more than one other purple offspring at a time (or in close proximity) becomes negligible. Further, the purple branching rate is given by the limit of the original branching rate weighted by the probability of survival, that is

$\lim_{T \rightarrow \infty} T\beta(1 - p_{T(1-s)})$, and this rate corresponds to a simple deterministic time change of a Yule tree in all near-critical cases.

An anonymous referee pointed out to us that Theorem 2.2 of [22] gives an incorrect formula in place of our (4), although the main Theorem 2.3 of [22] is nevertheless correct.

4. Spines and changes of measure. In this section we lay down many of the technical tools that we will need to prove the results in the previous sections. Our two most important signposts will be Proposition 7, which translates questions about uniformly chosen particles under \mathbb{P} into calculations under a new measure \mathbb{Q} ; and Proposition 16, which is a version of Campbell's formula under \mathbb{Q} which will be central to our analysis.

First we must introduce \mathbb{Q} , and we begin by describing the idea of *spines*, which introduce extra information into our tree by allocating *marks* to certain special particles. Spine methods are well known; for a thorough treatment see [13]. We give only a brief introduction.

4.1. *The k -spine measure \mathbb{P}^k .* We define a new measure \mathbb{P}^k under which there are k distinguished lines of descent, which we call spines. Briefly, \mathbb{P}^k is simply an extension of \mathbb{P} in that all particles behave as in the original branching process; the only difference is that some particles carry marks showing that they are part of a spine.

Under \mathbb{P}^k particles behave as follows:

- We begin with one particle which carries k marks $1, 2, \dots, k$.
- We think of each of the marks $1, \dots, k$ as distinguishing a particular line of descent or “spine”, and define ξ_t^i to be the label of whichever particle carries mark i at time t .
- A particle carrying j marks $b_1 < b_2 < \dots < b_j$ at time t branches at rate r , dying and being replaced by a random number of particles according to the law of L , independently of the rest of the system, just as under \mathbb{P} .
- Given that a particles v_1, \dots, v_a are born at a branching event as above, the j marks each choose a particle to follow independently and uniformly at random from amongst the a available. Thus for each $1 \leq l \leq a$ and $1 \leq i \leq j$ the probability that v_l carries mark b_i just after the branching event is $1/a$, independently of all other marks.
- If a particle carrying $j > 0$ marks $b_1 < b_2 < \dots < b_j$ dies and is replaced by 0 particles, then its marks are transferred to the graveyard Δ .

Again we emphasise that under \mathbb{P}^k , the system behaves exactly as under \mathbb{P} except that some particles carry extra marks showing the lines of descent of k spines. We write $\xi_t = (\xi_t^1, \dots, \xi_t^k)$. Obviously ξ_t depends on k too, but we omit this from the notation.

We let n_t be the number of distinct spines (i.e. the number of particles carrying marks) at time t , and for $i \geq 1$

$$\psi_i = \inf\{t \geq 0 : n_t \notin \{1, \dots, i\}\}$$

with $\psi_0 = 0$. We view ψ_i as the i th spine split time (although, for example, the first and second spine split times may be equal—corresponding to marks following three different particles at the first branching event). We also let ρ_t^i be the number of marks following spine i .

The set of distinct spine particles at any time t , and the marks that are following those spine particles, induce a partition \mathcal{Z}_t^k of $\{1, \dots, k\}$. That is, i and j are in the same block of \mathcal{Z}_t^k if $\xi_t^i = \xi_t^j$. If we then let $Z_t^k = \mathcal{Z}_{\psi_t}^k$ for $i = 0, \dots, k-1$, we have created a discrete collection of partitions Z_0, Z_1, \dots, Z_{k-1} which describe the topological information about the spines without the information about the spine split times. It will occasionally be useful to use the σ -algebra $\mathcal{H}' = \sigma(Z_0, Z_1, \dots)$.

For any particle $u \in \mathcal{N}_t$, there exists a last time at which u was a spine (which may be t). If this time equals ψ_i for some i , then we say that u is a *residue* particle; if it does not equal ψ_i for any i , and u is not a spine, then we say that u is *ordinary*. Each particle is exactly one of residue, ordinary, or a spine.

Of course \mathbb{P}^k is not defined on the same σ -algebra as \mathbb{P} . We let \mathcal{F}_t^k be the filtration containing all information about the system, including the k spines, up to time t ; then \mathbb{P}^k is defined on \mathcal{F}_∞^k . For more details see [13, Section 5]. Let \mathcal{F}_t^0 be the filtration containing only the information about the Galton-Watson tree. Let $\tilde{\mathcal{G}}_t^k$ be the filtration containing all the information about the k spines (including the birth events along the k spines) up to time t , but none of the information about the rest of the tree. Finally let \mathcal{G}_t^k be the filtration containing information only about spine splitting events (including which marks follow which spines); \mathcal{G}_t^k does not know when births of ordinary particles from the spines occur.

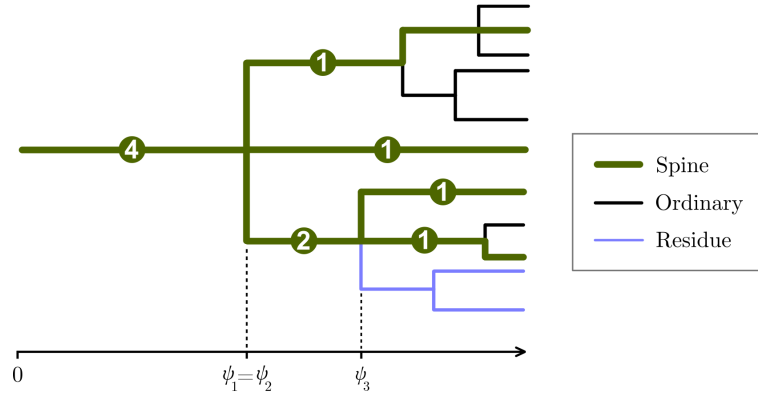


Fig 4: Spines, ordinary particles and residue particles. The horizontal axis represents time. The numbers show how many marks are carried by each spine.

4.2. A change of measure. We will now introduce a new measure. Under this measure, the k spines will be uniformly chosen (without replacement) at time T , which will allow us to represent uniformly chosen particles under \mathbb{P} as calculations using the spines under our new measure. This very natural new measure has some remarkable properties, including the fact that it can be fully described forwards in time. Without this new measure we found calculating with uniformly chosen particles to be intractable.

Throughout the rest of this section we fix $k \geq 1$ and assume that $\mathbb{P}[L^k] < \infty$. This condition will be relaxed later (see (20)), but for now it is required even to define our change of measure.

For any set S and $k \geq 1$, let $S^{(k)}$ be the set of distinct k -tuples from S , and for $n \geq 0$, write

$$n^{(k)} = \begin{cases} n(n-1)(n-2)\dots(n-k+1) & \text{if } n \geq k \\ 0 & \text{otherwise.} \end{cases}$$

Note that $|S^{(k)}| = |S|^{(k)}$. For $t \geq 0$, define

$$g_{k,t} := \mathbb{1}_{\{\xi_t^i \neq \xi_t^j \forall i \neq j\}} \prod_{i=1}^k \prod_{v < \xi_t^i} L_v \quad \text{and} \quad \zeta_{k,t} := \frac{g_{k,t}}{\mathbb{P}[N_t^{(k)}]}.$$

LEMMA 6. *For any $t \geq 0$, $\mathbb{P}^k[g_{k,t} | \mathcal{F}_t^0] = N_t^{(k)}$. In particular, $\mathbb{P}^k[\zeta_{k,t}] = 1$.*

PROOF. From the definition of $g_{k,t}$,

$$\mathbb{P}^k[g_{k,t}|\mathcal{F}_t^0] = \mathbb{P}^k\left[\sum_{u \in \mathcal{N}_t^{(k)}} \mathbb{1}_{\{\xi_t = u\}} \prod_{i=1}^k \prod_{v < u_i} L_v \mid \mathcal{F}_t^0\right] = \sum_{u \in \mathcal{N}_t^{(k)}} \left(\prod_{i=1}^k \prod_{v < u_i} L_v\right) \mathbb{P}^k(\xi_t = u \mid \mathcal{F}_t^0).$$

Recall that the marks act independently, and at each branching event choose uniformly amongst the available children. Therefore

$$(5) \quad \mathbb{P}^k(\xi_t = u \mid \mathcal{F}_t^0) = \prod_{i=1}^k \mathbb{P}^k(\xi_t^i = u_i \mid \mathcal{F}_t^0) = \prod_{i=1}^k \prod_{v < u_i} \frac{1}{L_v}.$$

Thus

$$\mathbb{P}^k[g_{k,t}|\mathcal{F}_t^0] = \sum_{u \in \mathcal{N}_t^{(k)}} 1 = |\mathcal{N}_t^{(k)}| = N_t^{(k)}.$$

This gives the first part of the result, and taking expectations gives the second. \square

We now fix $T > 0$ and define a new probability measure $\mathbb{Q}^{k,T}$ by setting

$$(6) \quad \frac{d\mathbb{Q}^{k,T}}{d\mathbb{P}^k} \Big|_{\mathcal{F}_T^k} := \frac{\mathbb{1}_{\{\xi_T^i \neq \xi_T^j \ \forall i \neq j\}} \prod_{i=1}^k \prod_{v < \xi_T^i} L_v}{\mathbb{P}[N_T(N_T - 1) \dots (N_T - k + 1)]} = \zeta_{k,T}$$

Often, when the choice of T and k is clear, we write \mathbb{P} instead of \mathbb{P}^k (since \mathbb{P}^k is an extension of \mathbb{P} this should not cause any problems) and \mathbb{Q} instead of $\mathbb{Q}^{k,T}$. Then, by Lemma 6,

$$(7) \quad \frac{d\mathbb{Q}^{k,T}}{d\mathbb{P}^k} \Big|_{\mathcal{F}_T^0} = \frac{N_T(N_T - 1) \dots (N_T - k + 1)}{\mathbb{P}[N_T(N_T - 1) \dots (N_T - k + 1)]} = \frac{N_T^{(k)}}{\mathbb{P}[N_T^{(k)}]} =: Z_{k,T}.$$

To see why the measure $\mathbb{Q}^{k,T}$ will be useful to us, we show how questions about particles sampled uniformly without replacement under \mathbb{P} become questions about the spines under \mathbb{Q} .

PROPOSITION 7. *Suppose that f is a measurable functional on the genealogies of k -tuples of particles. Then*

$$\mathbb{P}\left[\frac{1}{N_T^{(k)}} \sum_{u \in \mathcal{N}_T^{(k)}} f(u) \mid N_T \geq k\right] = \frac{\mathbb{P}[N_T^{(k)}]}{\mathbb{P}(N_T \geq k)(k-1)!} \int_0^\infty (e^z - 1)^{k-1} \mathbb{Q}^{k,T}\left[e^{-zN_T} f(\xi_T)\right] dz.$$

We defer the proof of this result to section 4.6.

4.3. Description of $\mathbb{Q}^{k,T}$. In this section, we give a full description of the measure $\mathbb{Q}^{k,T}$. We defer the proofs to section 4.5.

Our first lemma states that $\mathbb{Q}^{k,T}$ satisfies a time-dependent Markov branching property, in that the descendants of any particle behave independently of the rest of the tree.

LEMMA 8 (Symmetry lemma). *Suppose that $v \in \mathcal{N}_t$ is carrying j marks at time t . Then, under $\mathbb{Q}^{k,T}$, the subtree generated by v after time t is independent of the rest of the system and behaves as if under $\mathbb{Q}^{j,T-t}$.*

We will see in Section 4.6 (specifically (11) and the discussion following it) that particles that are not spines behave exactly as under \mathbb{P}^k : they branch at rate r and have offspring distribution L . The behaviour of the spine particles is more complicated.

Recall that τ_\emptyset is the first branching event, and ψ_1 is the time of the first spine splitting event, i.e.

$$\psi_1 = \inf\{t \geq 0 : \exists i, j \text{ with } \xi_t^i \neq \xi_t^j\}.$$

(Note that if the spines die without giving birth to any children, this counts as a splitting event.) By the symmetry lemma, in order to understand the split times under \mathbb{Q} , it suffices to understand the distributions of τ_\emptyset and ψ_1 .

LEMMA 9. *For any $t \in [0, T]$ and $k \geq 0$, we have*

$$\mathbb{Q}^{k,T}(\tau_\phi > t) = \frac{\mathbb{P}^k[N_{T-t}^{(k)}]}{\mathbb{P}^k[N_T^{(k)}]} e^{-rt}, \quad \mathbb{Q}^{k,T}(\psi_1 > t) = \frac{\mathbb{P}^k[N_{T-t}^{(k)}]}{\mathbb{P}^k[N_T^{(k)}]} e^{(m-1)rt}$$

and $\mathbb{Q}^{k,T}(\tau_\phi > t | \psi_1 > t) = e^{-mrt}$.

The third part of Lemma 9 and the symmetry lemma tell us that given \mathcal{G}_T^k (the information about spine splitting events), under $\mathbb{Q}^{k,T}$ each spine gives birth to non-spine particles according to a Poisson process of rate mr , independently of everything else. In particular when there are n distinct spines alive, there are n independent Poisson point processes and the total rate at which non-spine particles are immigrated along the spines is nmr .

We call birth events that occur along the spines, but which do not occur at spine splitting events, *births off the spine*. The following lemma tells us the distribution of the number of children born at such events.

LEMMA 10. *For any $j \geq 0$, $k \geq 1$ and $0 \leq t < T$, $\mathbb{Q}^{k,T}(L_\emptyset = j | \tau_\emptyset = t, \psi_1 > t) = \frac{jp_j}{m}$.*

A random variable that takes the value j with probability jp_j/m for each j is said to be *size-biased* (relative to L). Lemma 10 then tells us (in conjunction with the symmetry lemma) that births off any spine are always size-biased, no matter how many marks are following that particular spine. (The number of marks therefore only affects spine splitting events.)

To have a complete description of the behaviour of the process under $\mathbb{Q}^{k,T}$, it remains to understand how the marks distribute themselves amongst the available children at a spine splitting event. To do this, we write \mathcal{P}_t^ξ for the partition of $\{1, \dots, k\}$ induced by letting i and j be in the same block if the i th and j th spines are following the same particle at time t . By the symmetry lemma, again it suffices to consider the first spine splitting event.

LEMMA 11. *Conditional on $\{\psi_1 > t\}$, the $\mathbb{Q}^{k,T}$ -conditional probability that during the time interval $[t, t+h)$, the spine particle dies and gives birth to l offspring, and at this time the marks are partitioned according to a partition P with blocks of sizes a_1, \dots, a_n , is given by*

$$\mathbb{Q}^{k,T}(\psi_1 < t+h, \mathcal{P}_{\psi_1}^\xi = P, L_{\xi_t^1} = l \mid \psi_1 > t) = p_l l^{(n)} \frac{\prod_{i=1}^n \mathbb{P}^k[N_{T-t}^{(a_i)}]}{\mathbb{P}^k[N_{T-t}^{(k)}]} (rh + o(h)).$$

For a collection of positive integers a_1, \dots, a_n whose sum is k , write $n_j = \#\{i : a_i = j\}$ for each $j \geq 1$. (Note that $\sum_{j=1}^k n_j = n$ and $\sum_{j=1}^k j n_j = k$.) Then the number of partitions of $\{1, \dots, k\}$

into blocks of sizes a_1, \dots, a_n is

$$\frac{k!}{\prod_{i=1}^n a_i!} \frac{1}{\prod_{j=1}^k n_j!}$$

Combining this observation with Lemmas 9 and 11 gives us the following corollary.

COROLLARY 12.

$$\begin{aligned} \mathbb{Q}^{k,T} \left(\psi_1 \in [t, t+dt), \text{ spines split into groups of sizes } a_1, \dots, a_n, L_{\xi_t^1} = l \right) \\ = \frac{l^{(n)} p_l}{\mathbb{P}[L^{(n)}]} \frac{k!}{\prod_{i=1}^n a_i! \prod_{j=1}^{k-1} n_j!} \mathbb{P}[L^{(n)}] r e^{(m-1)rt} \frac{\prod_{i=1}^n \mathbb{P}^k[N_{T-t}^{(a_i)}]}{\mathbb{P}^k[N_T^{(k)}]} dt. \end{aligned}$$

4.4. *Understanding the measure $\mathbb{Q}^{k,T}$ as $T \rightarrow \infty$.* To help the reader to understand the results from the previous section, particularly Corollary 12, we let $T \rightarrow \infty$ and ask what happens to the tree drawn out by the spines. For brevity we will concentrate on the critical case $m = 1$, although similar calculations could be done in near-critical cases. Take $m = 1$, $n = 2$ and $t = sT$ in Corollary 12; if $a_1 \neq a_2$ then we get

$$\begin{aligned} \mathbb{Q}^{k,T} \left(\psi_1 \in [sT, sT + Tds), \text{ spines split into two groups of sizes } a_1, a_2, L_{\xi_{sT}^1} = l \right) \\ = \frac{l(l-1)p_l}{\mathbb{P}[L(L-1)]} \frac{k!}{a_1! a_2!} \mathbb{P}[L(L-1)] r \frac{\mathbb{P}^k[N_{T(1-s)}^{(a_1)}] \mathbb{P}^k[N_{T(1-s)}^{(a_2)}]}{\mathbb{P}^k[N_T^{(k)}]} Tds. \end{aligned}$$

We now let $T \rightarrow \infty$ and use Kolmogorov's theorem that $T\mathbb{P}(N_{uT} > 0) \rightarrow 2/(\sigma^2 r u)$, as well as Yaglom's theorem which says that conditional on survival, N_{uT}/T converges in distribution to an exponential random variable of rate $2/(\sigma^2 r u)$. Letting \mathcal{E}_1 and \mathcal{E}_2 be exponentially distributed with parameters $2/(\sigma^2 r(1-s))$ and $2/(\sigma^2 r)$, respectively, this gives

$$\begin{aligned} \lim_{T \rightarrow \infty} \mathbb{Q}^{k,T} \left(\psi_1 \in [sT, sT + Tds), \text{ spines split into two groups of sizes } a_1, a_2, L_{\xi_{sT}^1} = l \right) \\ = \frac{l(l-1)p_l}{\mathbb{P}[L(L-1)]} \frac{k!}{a_1! a_2!} \mathbb{P}[L(L-1)] r \frac{\frac{2}{T\sigma^2 r(1-s)} T^{a_1} \mathbb{P}^k[\mathcal{E}_1^{a_1}] \frac{2}{T\sigma^2 r(1-s)} T^{a_2} \mathbb{P}^k[\mathcal{E}_1^{a_2}]}{\frac{2}{T\sigma^2 r} T^k \mathbb{P}^k[\mathcal{E}_2^k]} Tds \\ = l(l-1)p_l r \frac{(\sigma^2 r(1-s)/2)^{a_1-1} (\sigma^2 r(1-s)/2)^{a_2-1}}{(\sigma^2 r/2)^{k-1}} ds = l(l-1)p_l \frac{2}{\sigma^2} (1-s)^{k-2} ds. \end{aligned}$$

If $a_1 = a_2$ then there is an extra factor of $1/2$ as the two blocks can be re-ordered.

As there are $k-1$ possible (ordered) ways of splitting k into two groups of non-zero size, and from the above each of these ways is equally likely,

$$\begin{aligned} \lim_{T \rightarrow \infty} \mathbb{Q}^{k,T} \left(\psi_1 \in [sT, sT + Tds), \text{ spines split into two groups, } L_{\xi_{sT}^1} = l \right) \\ = \frac{l(l-1)}{\sigma^2} p_l (k-1) (1-s)^{k-2} ds. \end{aligned}$$

We note that if we sum the above quantity over l and integrate over $s \in [0, 1]$ we obtain 1. This means that, in the limit as $T \rightarrow \infty$, at the first spine splitting event ψ_1 , the k spines always split into exactly two groups. We also see that the number of spines in each of the groups is uniform on $\{1, \dots, k-1\}$, and the total number of offspring at this time is doubly-size-biased. Finally, the first

splitting time, when rescaled by $1/T$, converges in distribution to the minimum of k independent uniform random variables on $[0, 1]$.

The symmetry lemma, Lemma 8, tells us that we can extend our understanding of the *first* spine splitting event to all spine splitting events. When a collection of spines decides to split, they always (in the limit as $T \rightarrow \infty$) split uniformly into two groups; this property is shared by the tree drawn out by the Kingman coalescent. Furthermore the $k - 1$ spine split times, when rescaled by $1/T$, are independent and uniformly distributed on $[0, 1]$.

We stress again that this is true only in the critical case; if instead we are in the near-critical case when $m = 1 + \mu/T + o(1/T)$ (see Section 2.2) then the uniform density for the independent split times is replaced by $\frac{r\mu e^{r\mu s}}{e^{r\mu} - 1} ds$. In particular, the near-critical case is simply a deterministic time-change of the critical picture.

4.5. *Proofs of properties of $\mathbb{Q}^{k,T}$.* The following simple general lemma will be useful.

LEMMA 13. *Suppose that μ and ν are probability measures on the σ -algebra \mathcal{F} , and that \mathcal{G} is a sub- σ -algebra of \mathcal{F} . If*

$$\left. \frac{d\mu}{d\nu} \right|_{\mathcal{F}} = Y \quad \text{and} \quad \left. \frac{d\mu}{d\nu} \right|_{\mathcal{G}} = Z,$$

then for any non-negative random variable X , $Z\mu[X|\mathcal{G}] = \nu[XY|\mathcal{G}]$, ν -almost surely.

PROOF. For any $A \in \mathcal{G}$, $\nu[XY\mathbb{1}_A] = \mu[X\mathbb{1}_A] = \mu[\mu[X|\mathcal{G}]\mathbb{1}_A] = \nu[Z\mu[X|\mathcal{G}]\mathbb{1}_A]$. Since $Z\mu[X|\mathcal{G}]$ is \mathcal{G} -measurable, it therefore satisfies the definition of conditional expectation of XY with respect to \mathcal{G} under ν . \square

We can now prove the symmetry lemma.

PROOF OF LEMMA 8. Fix t, T and v . Let \mathcal{H} be the σ -algebra generated by all the information except in the subtree generated by v after time t . Then it suffices to show that for $s \in (t, T]$ and $i \geq 0$,

$$\mathbb{Q}^{k,T}(\tau_v > s, L_v = i | \mathcal{H}) = \mathbb{Q}^{j,T-t}(\tau_\emptyset > s - t, L_\emptyset = i)$$

almost surely. Recall that

$$g_{k,T} = \mathbb{1}_{\{\xi_T^i \neq \xi_T^j \forall i \neq j\}} \prod_{i=1}^k \prod_{v < \xi_T^i} L_v \quad \text{and} \quad \zeta_{k,T} = \frac{g_{k,T}}{\mathbb{P}[N_T^{(k)}]}.$$

Let I be the set of marks carried by v at time t , and let

$$\tilde{g} = \mathbb{1}_{\{\xi_T^i \neq \xi_T^j \forall i \neq j, i, j \in I^c\}} \prod_{i \in I} \prod_{\xi_t^i \leq v < \xi_T^i} L_v \quad \text{and} \quad h = \mathbb{1}_{\{\xi_T^i \neq \xi_T^j \forall i \neq j, i, j \in I^c\}} \left(\prod_{i \notin I} \prod_{v < \xi_T^i} L_v \right) \prod_{i \in I} \prod_{v < \xi_t^i} L_v.$$

Note that h is \mathcal{H} -measurable and $g_{k,T} = \tilde{g}h$.

By Lemma 13, $\mathbb{Q}^{k,T}$ -almost surely,

$$\mathbb{Q}^{k,T}(\tau_v > s, L_v = i | \mathcal{H}) = \frac{1}{\mathbb{P}^k[\zeta_{k,T} | \mathcal{H}]} \mathbb{P}^k[\zeta_{k,T} \mathbb{1}_{\{\tau_v > s, L_v = i\}} | \mathcal{H}].$$

Cancelling factors of $\mathbb{P}^k[N_T^{(k)}]$ and using the fact that $g_{k,T} = \tilde{g}h$ where h is \mathcal{H} -measurable, we get

$$\mathbb{Q}^{k,T}(\tau_v > s, L_v = i | \mathcal{H}) = \frac{1}{h \mathbb{P}^k[\tilde{g} | \mathcal{H}]} h \mathbb{P}^k[\tilde{g} \mathbb{1}_{\{\tau_v > s, L_v = i\}} | \mathcal{H}] = \frac{\mathbb{P}^k[\tilde{g} \mathbb{1}_{\{\tau_v > s, L_v = i\}} | \mathcal{H}]}{\mathbb{P}^k[\tilde{g} | \mathcal{H}]}.$$

By the Markov branching property under \mathbb{P}^k , the behaviour of the subtree generated by v after time t is independent of the rest of the system and—on the event that v is carrying j marks at time t —behaves as if under \mathbb{P}^j . Thus

$$\mathbb{Q}^{k,T}(\tau_v > s, L_v = i | \mathcal{H}) = \frac{\mathbb{P}^j[g_{j,T-t} \mathbb{1}_{\{\tau_\emptyset > s-t, L_\emptyset = i\}}]}{\mathbb{P}^j[g_{j,T-t}]}$$

almost surely. Applying Lemma 6 establishes the result. \square

Next we prove Lemma 9, which gives the distribution of the split times under $\mathbb{Q}^{k,T}$.

PROOF OF LEMMA 9. For the first statement,

$$\mathbb{Q}(\tau_\emptyset > t) = \mathbb{P}[\zeta_{k,T} \mathbb{1}_{\{\tau_\emptyset > t\}}] = \frac{1}{\mathbb{P}[N_T^{(k)}]} \mathbb{P}[g_{k,T} \mathbb{1}_{\{\tau_\emptyset > t\}}].$$

By the Markov property and Lemma 6,

$$\mathbb{P}[g_{k,T} \mathbb{1}_{\{\tau_\emptyset > t\}}] = \mathbb{P}(\tau_\emptyset > t) \mathbb{P}[g_{k,T-t}] = e^{-rt} \mathbb{P}[N_{T-t}^{(k)}]$$

as required. For the second statement,

$$\mathbb{Q}(\psi_1 > t) = \mathbb{P}[\zeta_{k,T} \mathbb{1}_{\{\psi_1 > t\}}] = \frac{1}{\mathbb{P}[N_T^{(k)}]} \mathbb{P}[g_{k,T} \mathbb{1}_{\{\psi_1 > t\}}],$$

and by the Markov property and Lemma 6,

$$\mathbb{P}[g_{k,T} \mathbb{1}_{\{\psi_1 > t\}}] = \mathbb{P}\left[\left(\prod_{v < \xi_t^1} L_v^k\right) \mathbb{1}_{\{\psi_1 > t\}}\right] \mathbb{P}[g_{k,T-t}] = \mathbb{P}\left[\left(\prod_{v < \xi_t^1} L_v^k\right) \mathbb{1}_{\{\psi_1 > t\}}\right] \mathbb{P}[N_{T-t}^{(k)}].$$

Putting these two lines together we get

$$(8) \quad \mathbb{Q}(\psi_1 > t) = \frac{\mathbb{P}^k[N_{T-t}^{(k)}]}{\mathbb{P}^k[N_T^{(k)}]} \mathbb{P}\left[\left(\prod_{v < \xi_t^1} L_v^k\right) \mathbb{1}_{\{\psi_1 > t\}}\right].$$

Note that $\psi > t$ if and only if all k marks are following the same particle at time t (which must also be alive); thus

$$\mathbb{P}\left[\left(\prod_{v < \xi_t^1} L_v^k\right) \mathbb{1}_{\{\psi_1 > t\}}\right] = \mathbb{P}\left[\sum_{u \in \mathcal{N}_t} \left(\prod_{v < u} L_v^k\right) \mathbb{1}_{\{\xi_t^1 = \dots = \xi_t^k = u\}}\right] = \mathbb{P}\left[\sum_{u \in \mathcal{N}_t} 1\right] = \mathbb{P}[N_t] = e^{(m-1)t}.$$

Substituting into (8) gives the result. The third statement follows easily from the first two. \square

We next prove Lemma 10, which says that births off the spine are size-biased.

PROOF OF LEMMA 10. From the definition of \mathbb{Q} ,

$$\begin{aligned} \mathbb{Q}(L_\emptyset = j | \tau_\emptyset = t, \psi_1 > t) &= \frac{\mathbb{P}[\zeta_{k,T} \mathbb{1}_{\{L_\emptyset = j\}} | \tau_\emptyset = t, \psi_1 > t]}{\mathbb{P}[\zeta_{k,T} | \tau_\emptyset = t, \psi_1 > t]} \\ &= \frac{\mathbb{P}[g_{k,T} \mathbb{1}_{\{L_\emptyset = j\}} | \tau_\emptyset = t, \psi_1 > t]}{\mathbb{P}[g_{k,T} | \tau_\emptyset = t, \psi_1 > t]} = \frac{\mathbb{P}[g_{k,T} \mathbb{1}_{\{L_\emptyset = j, \psi_1 > t\}} | \tau_\emptyset = t]}{\mathbb{P}[g_{k,T} \mathbb{1}_{\{\psi_1 > t\}} | \tau_\emptyset = t]}. \end{aligned}$$

If the first particle has i offspring, then the product appearing in the definition of $g_{k,T}$ sees a factor of i^k ; and the probability that all k spines follow the same one of these offspring is $1/i^{k-1}$. Thus, by the Markov property, for any i ,

$$\mathbb{P}[g_{k,T} \mathbb{1}_{\{L_\emptyset=i, \psi_1>t\}} | \tau_\emptyset = t] = p_i i^k \frac{1}{i^{k-1}} \mathbb{P}[g_{k,T-t}] = ip_i \mathbb{P}[g_{k,T-t}].$$

Thus

$$\mathbb{Q}(L_\emptyset = j | \tau_\emptyset = t, \psi_1 > t) = \frac{jp_j \mathbb{P}[g_{k,T-t}]}{\sum_i ip_i \mathbb{P}[g_{k,T-t}]} = \frac{jp_j}{m}. \quad \square$$

The final proof in this section is of Lemma 11, which completed the description of $\mathbb{Q}^{k,T}$.

PROOF OF LEMMA 11. By the symmetry lemma, for any $h \in (0, T-t]$,

$$\mathbb{Q}^{k,T}(\psi_1 < t+h, \mathcal{P}_{\psi_1}^\xi = P, L_{\xi_t^1} = l \mid \psi_1 > t) = \mathbb{Q}^{k,T-t}(\psi_1 < h, \mathcal{P}_{\psi_1}^\xi = P, L_\emptyset = l).$$

By the definition of $\mathbb{Q}^{k,T-t}$, this is equal to

$$\begin{aligned} (9) \quad & \frac{1}{\mathbb{P}^k[N_{T-t}^{(k)}]} \mathbb{P}^k \left[g_{k,T-t} \mathbb{1}_{\left\{ \psi_1 < h, \mathcal{P}_{\psi_1}^\xi = P, L_\emptyset = l \right\}} \right] \\ &= \frac{1}{\mathbb{P}^k[N_{T-t}^{(k)}]} \mathbb{P}^k(\psi_1 < h, \mathcal{P}_{\psi_1}^\xi = P, L_\emptyset = l) \mathbb{P}^k[g_{k,T-t} | \psi_1 < h, \mathcal{P}_{\psi_1}^\xi = P, L_\emptyset = l] \end{aligned}$$

First we consider

$$\begin{aligned} \mathbb{P}^k(\psi_1 < h, \mathcal{P}_{\psi_1}^\xi = P, L_\emptyset = l) &= \mathbb{P}^k(\psi_1 < h, L_\emptyset = l) \mathbb{P}^k(\mathcal{P}_{\psi_1}^\xi = P | \psi_1 < h, L_\emptyset = l) \\ &= \frac{l^{(n)}}{l^k} \mathbb{P}^k(\psi_1 < h, L_\emptyset = l) \end{aligned}$$

since $l^{(n)}/l^k$ is the probability that k balls put uniformly and independently into l bins give rise to the partition P .

Next we consider

$$\mathbb{P}^k[g_{k,T-t} | \psi_1 < h, \mathcal{P}_{\psi_1}^\xi = P, L_\emptyset = l].$$

Note that on the event $\{\mathcal{P}_{\psi_1}^\xi = P, L_\emptyset = l\}$, we have

$$g_{k,T-t} = \mathbb{1}_{\{\xi_{T-t}^i \neq \xi_{T-t}^j \forall i \neq j\}} \prod_{i=1}^k \prod_{v < \xi_{T-t}^i} L_v = l^k \prod_{p \in P} \mathbb{1}_{\{\xi_{T-t}^i \neq \xi_{T-t}^j \forall i \neq j \in p\}} \prod_{i \in p} \prod_{\xi_{\psi_1}^i \leq v < \xi_{T-t}^i} L_v.$$

Lemma 6 tells us that for each $p \in P$, on the event $\{\mathcal{P}_{\psi_1}^\xi = P, L_\emptyset = l\}$,

$$\mathbb{P}^k \left[\mathbb{1}_{\{\xi_{T-t}^i \neq \xi_{T-t}^j \forall i \neq j \in p\}} \prod_{i \in p} \prod_{\xi_{\psi_1}^i \leq v < \xi_{T-t}^i} L_v \mid \mathcal{F}_{\psi_1}^k \right] = \mathbb{P}^k[N_{T-t-u}^{(|p|)}]_{u=\psi_1}$$

On the event $\psi_1 < h$, we have

$$\mathbb{P}^k[N_{T-t-u}^{(|p|)}]_{u=\psi_1} = \mathbb{P}^k[N_{T-t}^{(|p|)}] + o(1)$$

and therefore

$$\mathbb{P}^k[g_{k,T-t} | \psi_1 < h, \mathcal{P}_{\psi_1}^\xi = P, L_\emptyset = l] = l^k \prod_{i=1}^n \mathbb{P}^k[N_{T-t}^{(a_i)}] + o(1).$$

Putting these calculations back into (9), we have shown that

$$\begin{aligned} & \mathbb{Q}^{k,T}(\psi_1 < t + h, \mathcal{P}_{\psi_1}^\xi = P, L_{\xi_t^1} = l \mid \psi_1 > t) \\ &= \frac{1}{\mathbb{P}^k[N_{T-t}^{(k)}]} (rh + o(h)) p_l \frac{l^{(n)}}{l^k} l^k \left(\prod_{i=1}^n \mathbb{P}^k[N_{T-t}^{(a_i)}] + o(1) \right) = p_l l^{(n)} \frac{\prod_{i=1}^n \mathbb{P}^k[N_{T-t}^{(a_i)}]}{\mathbb{P}^k[N_{T-t}^{(k)}]} (rh + o(h)) \end{aligned}$$

which completes the proof. \square

4.6. *Proof of Proposition 7.* Before we prove Proposition 7, we develop several partial results along the way.

Applying Lemma 13, we get that for any non-negative \mathcal{F}_T^k -measurable random variable X , on the event $Z_{k,T} > 0$,

$$(10) \quad \mathbb{Q}^{k,T}[X | \mathcal{F}_T^0] = \frac{1}{Z_{k,T}} \mathbb{P}^k[X \zeta_{k,T} | \mathcal{F}_T^0],$$

and on the event $\zeta_{k,T} > 0$, since $\zeta_{k,T}$ is $\tilde{\mathcal{G}}_T^k$ -measurable,

$$(11) \quad \mathbb{Q}^{k,T}[X | \tilde{\mathcal{G}}_T^k] = \frac{1}{\zeta_{k,T}} \mathbb{P}^k[X \zeta_{k,T} | \tilde{\mathcal{G}}_T^k] = \mathbb{P}^k[X | \tilde{\mathcal{G}}_T^k].$$

This last equation (11) tells us in particular that any event that is independent of $\tilde{\mathcal{G}}_T^k$ under \mathbb{P} has the same probability under \mathbb{Q} as it does under \mathbb{P} . In other words, non-spine particles behave under \mathbb{Q} exactly as they do under \mathbb{P} : they branch at rate r and have offspring distribution L .

Also note that under $\mathbb{Q}^{k,T}$, the k spine particles are almost surely distinct at time T , since directly from the definition of $\zeta_{k,T}$, $\mathbb{Q}^{k,T}(\exists i \neq j : \xi_T^i = \xi_T^j) = \mathbb{P}[\zeta_{k,T} \mathbb{1}_{\{\exists i \neq j : \xi_T^i = \xi_T^j\}}] = 0$.

In fact, the next lemma tells us that under $\mathbb{Q}^{k,T}$, the spines are chosen uniformly without replacement from those alive at time T .

LEMMA 14. *For any $u \in \mathcal{N}_T^{(k)}$, on the event $N_T \geq k$, we have $\mathbb{Q}^{k,T}(\xi_T = u | \mathcal{F}_T^0) = 1/N_T^{(k)}$.*

PROOF. Note that if $N_T \geq k$ then $Z_{k,T} > 0$. Then by (10), for any $u \in \mathcal{N}_T^{(k)}$,

$$\mathbb{Q}(\xi_T = u | \mathcal{F}_T^0) = \frac{1}{Z_{k,T}} \mathbb{P}[\zeta_{k,T} \mathbb{1}_{\{\xi_T = u\}} | \mathcal{F}_T^0] = \frac{\mathbb{P}[N_T^{(k)}]}{N_T^{(k)}} \frac{1}{\mathbb{P}[N_T^{(k)}]} \left(\prod_{i=1}^k \prod_{v < u_i} L_v \right) \mathbb{P}(\xi_T = u | \mathcal{F}_T^0).$$

The result now follows by applying (5). \square

As part of proving Proposition 7 we will need to calculate quantities like $\mathbb{Q}^{k,T}[1/N_T^{(k)} | \mathcal{G}_T^k]$. The next lemma allows us to work with moment generating functions, which are somewhat easier to deal with and will lead to an important product structure from the independent contributions to N_T along different branches of the k spines' genealogical tree under $\mathbb{Q}^{k,T}$.

LEMMA 15. *For any $k \in \mathbb{N}$ and positive integer valued random variable N under an expectation operator E , we have*

$$E \left[\frac{1}{N(N-1) \dots (N-k+1)} \right] = \frac{1}{(k-1)!} \int_0^\infty (e^z - 1)^{k-1} E[e^{-zN}] dz.$$

In particular, for any $k \in \mathbb{N}$ and $T \geq 0$,

$$\mathbb{Q}^{k,T} \left[\frac{1}{N_T^{(k)}} \middle| \mathcal{G}_T^k \right] = \frac{1}{(k-1)!} \int_0^\infty (e^z - 1)^{k-1} \mathbb{Q}^{k,T}[e^{-zN_T} | \mathcal{G}_T^k] dz.$$

PROOF. We show, by induction on j , that for all $j = 1, \dots, k$,

$$E \left[\frac{1}{N^{(j)}} \right] = \frac{1}{(j-1)!} \int_0^\infty (e^z - 1)^{j-1} E[e^{-zN}] dz.$$

The case $j = 1$ follows from Fubini's theorem. For the general step, observe that for $j \leq k-1$,

$$\int_0^\infty (e^z - 1)^j E[e^{-zN}] dz = \int_0^\infty (e^z - 1)^{j-1} E[e^{-z(N-1)}] dz - \int_0^\infty (e^z - 1)^{j-1} E[e^{-zN}] dz$$

and by the induction hypothesis, this equals

$$(j-1)! E \left[\frac{1}{(N-1)^{(j)}} \right] - (j-1)! E \left[\frac{1}{N^{(j)}} \right] = (j-1)! E \left[\frac{N - (N-j)}{N^{(j+1)}} \right] = j! E \left[\frac{1}{N^{(j+1)}} \right].$$

This gives the result. \square

We can now prove Proposition 7, which translates questions about particles sampled uniformly without replacement under \mathbb{P} into questions about the spines under \mathbb{Q} .

PROOF OF PROPOSITION 7. First note that

$$\mathbb{Q}[f(\xi_T) | \mathcal{F}_T^0] \mathbb{1}_{\{N_T \geq k\}} = \mathbb{Q} \left[\sum_{u \in \mathcal{N}_T^{(k)}} \mathbb{1}_{\{\xi_T = u\}} f(u) \middle| \mathcal{F}_T^0 \right] = \sum_{u \in \mathcal{N}_T^{(k)}} f(u) \mathbb{Q}(\xi_T = u | \mathcal{F}_T^0)$$

almost surely. Applying Lemma 14, we get

$$\mathbb{Q}[f(\xi_T) | \mathcal{F}_T^0] \mathbb{1}_{\{N_T \geq k\}} = \frac{1}{N_T^{(k)}} \sum_{u \in \mathcal{N}_T^{(k)}} f(u)$$

almost surely (where we take the right-hand side to be zero if $N_T < k$). Taking \mathbb{P} -expectations,

$$\mathbb{P} \left[\frac{1}{N_T^{(k)}} \sum_{u \in \mathcal{N}_T^{(k)}} f(u) \right] = \mathbb{P}[\mathbb{Q}[f(\xi_T) | \mathcal{F}_T^0] \mathbb{1}_{\{N_T \geq k\}}].$$

Applying (7) and recalling that under \mathbb{Q} there are at least k particles alive at time T ,

$$(12) \quad \mathbb{P} \left[\frac{1}{N_T^{(k)}} \sum_{u \in \mathcal{N}_T^{(k)}} f(u) \right] = \mathbb{Q} \left[\frac{1}{Z_{k,T}} \mathbb{Q}[f(\xi_T) | \mathcal{F}_T^0] \right] = \mathbb{Q} \left[\frac{f(\xi_T)}{Z_{k,T}} \right] = \mathbb{P}[N_T^{(k)}] \mathbb{Q} \left[\frac{f(\xi_T)}{N_T^{(k)}} \right].$$

Dividing through by $\mathbb{P}(N_T \geq k)$ and applying Lemma 15 gives the result. \square

4.7. *Campbell's formula.* One of the key elements that we need to carry out our calculations will be a version of Campbell's formula. Let \tilde{N}_t be the number of *ordinary* particles alive at time t —that is, they are not spines, and did not split from spines at spine splitting events. Recall that we also defined n_t to be the number of distinct spines alive at time t .

We write $F(\theta, t) = \mathbb{P}[\theta^{N_t}]$ and $u(\theta) = \mathbb{P}[\theta^L] - \theta$. These functions satisfy the Kolmogorov forwards and backwards equations

$$(13) \quad \frac{\partial}{\partial t} F(\theta, t) = ru(\theta) \frac{\partial}{\partial \theta} F(\theta, t) \quad \text{and} \quad \frac{\partial}{\partial t} F(\theta, t) = ru(F(\theta, t));$$

see [5, Chapter III, Section 3]. Our main aim is to show the following:

PROPOSITION 16. *For any $z \geq 0$, $\mathbb{Q}^{k,T}$ -almost surely.*

$$\mathbb{Q}^{k,T}[e^{-z\tilde{N}_T} | \mathcal{G}_T^k] = \prod_{i=0}^{k-1} \left(e^{-r(m-1)(T-\psi_i)} \frac{u(F(e^{-z}, T-\psi_i))}{u(e^{-z})} \right).$$

Notice in particular that the right-hand side depends only on the values of the split times $\psi_1, \dots, \psi_{k-1}$ of the spines, not any of the other information in \mathcal{G}_T^k (for example the topological information about the tree). This—used in conjunction with Proposition 7—is a large part of the reason that the split times of our k uniformly chosen particles are (asymptotically) independent of the topological information in the induced tree.

The main step in proving Proposition 16 comes from the next lemma.

LEMMA 17. *For any $z \geq 0$,*

$$\mathbb{Q}[e^{-z\tilde{N}_T} | \mathcal{G}_T^k] = \prod_{i=0}^{k-1} \exp \left(-r(m-1)(T-\psi_i) + r \int_0^{T-\psi_i} u'(\mathbb{P}[e^{-zN_s}]) ds \right).$$

$\mathbb{Q}^{k,T}$ -almost surely.

PROOF. Let Λ_T be the total number of birth events off the spines (i.e. births along spines that are not spine splitting events) before time T . Recall (from Lemma 9 and the symmetry lemma) that under $\mathbb{Q}^{k,T}$ each spine gives birth to non-spine particles according to a Poisson process of rate rm , independently of everything else. Thus at any time $s \in [0, T]$, the total rate at which spine particles give birth to non-spine particles is rmn_s . Besides, such births are size biased (by Lemma 10 and the symmetry lemma). Finally, once a particle is born off the spines, it generates a tree that behaves exactly as under \mathbb{P} (see (11) and the discussion that follows).

Thus, letting $\lambda_T = \int_0^T n_s ds$,

$$\mathbb{Q}[e^{-z\tilde{N}_T} | \mathcal{G}_T^k] = \sum_{j=0}^{\infty} \mathbb{Q}(\Lambda_T = j | \mathcal{G}_T^k) \left(\int_0^T \sum_{i=1}^{\infty} \frac{ip_i}{m} \mathbb{P}[e^{-zN_{T-s}}]^{i-1} \frac{n_s}{\lambda_T} ds \right)^j.$$

Since $\mathbb{Q}(\Lambda_T = j | \mathcal{G}_T^k) = e^{-rm\lambda_T} (rm\lambda_T)^j / j!$, we get

$$\mathbb{Q}[e^{-z\tilde{N}_T} | \mathcal{G}_T^k] = e^{-rm\lambda_T} \sum_{j=0}^{\infty} \frac{1}{j!} \left(r \int_0^T \sum_{i=1}^{\infty} ip_i \mathbb{P}[e^{-zN_{T-s}}]^{i-1} n_s ds \right)^j.$$

Note that $\sum_{i=1}^{\infty} i p_i \theta^{i-1} = \frac{d}{d\theta} \sum_{i=1}^{\infty} p_i \theta^i = u'(\theta) + 1$ and therefore

$$\mathbb{Q}[e^{-z\tilde{N}_T} | \mathcal{G}_T^k] = \exp \left(-r(m-1)\lambda_T + r \int_0^T u'(\mathbb{P}[e^{-zN_{T-s}}]) n_s ds \right).$$

Now, we know that between times ψ_{i-1} and ψ_i we have exactly i distinct spine particles. Thus

$$\mathbb{Q}[e^{-z\tilde{N}_T} | \mathcal{G}_T^k] = \prod_{i=0}^{k-1} \exp \left(-r(m-1)(T - \psi_i) + r \int_{\psi_i}^T u'(\mathbb{P}[e^{-zN_{T-s}}]) ds \right). \quad \square$$

PROOF OF PROPOSITION 16. Recalling (13) that $F(\theta, s)$ satisfies the backwards equation $\frac{\partial}{\partial s} F(\theta, s) = ru(F(\theta, s))$, by making the substitution $t = F(\theta, s)$ we see that

$$r \int_a^b u'(F(\theta, s)) ds = r \int_{F(\theta, a)}^{F(\theta, b)} \frac{u'(t)}{ru(t)} dt = \log \left(\frac{u(F(\theta, b))}{u(F(\theta, a))} \right).$$

Applying this to Lemma 17, we have

$$\mathbb{Q}[e^{-z\tilde{N}_T} | \mathcal{G}_T^k] = \prod_{i=0}^{k-1} \left(e^{-r(m-1)(T-\psi_i)} \frac{u(F(e^{-z}, T-\psi_i))}{u(F(e^{-z}, 0))} \right).$$

Noting that $F(e^{-z}, 0) = e^{-z}$ gives the result. \square

5. Birth-death processes. In this section we prove the results from Section 2.1. Recall the setup: fix $\alpha \geq 0$ and $\beta > 0$, and suppose that $r = \alpha + \beta$, $p_0 = \alpha/(\alpha + \beta)$ and $p_2 = \beta/(\alpha + \beta)$, with $p_j = 0$ for $j \neq 0, 2$. This is a birth-death process with birth rate β and death rate α . Since all particles have either 0 or 2 children, and under \mathbb{Q} the spines cannot have 0 children, they must always have 2 children. This simplifies the picture considerably.

5.1. *Elementary calculations with generating functions.* Suppose first that we are in the non-critical case $\alpha \neq \beta$. It is easy to calculate the moment generating function under \mathbb{P} for a birth-death process (see [5, Chapter III, Section 5]): for $\alpha \neq \beta$ and $\theta \in (0, 1)$,

$$F(\theta, t) := \mathbb{P}[\theta^{N_t}] = \frac{\alpha(1-\theta)e^{(\beta-\alpha)t} + \beta\theta - \alpha}{\beta(1-\theta)e^{(\beta-\alpha)t} + \beta\theta - \alpha}.$$

We then see that

$$\mathbb{P}(N_t = 0) = \lim_{\theta \downarrow 0} F(0, t) = \frac{\alpha e^{(\beta-\alpha)t} - \alpha}{\beta e^{(\beta-\alpha)t} - \alpha}.$$

Writing

$$p_t = \mathbb{P}(N_t = 0) = \frac{\alpha e^{(\beta-\alpha)t} - \alpha}{\beta e^{(\beta-\alpha)t} - \alpha} \quad \text{and} \quad q_t = \frac{\beta e^{(\beta-\alpha)t} - \beta}{\beta e^{(\beta-\alpha)t} - \alpha},$$

we get

$$F(\theta, t) = p_t + (1-p_t) \frac{(1-q_t)\theta}{1-q_t\theta} = p_t + (1-p_t)(1-q_t) \sum_{j=1}^{\infty} \theta^j q_t^{j-1}$$

and

$$\frac{\partial^k F(\theta, t)}{\partial \theta^k} = \frac{(1-p_t)(1-q_t)}{q_t} \frac{q_t^k k!}{(1-q_t\theta)^{k+1}}.$$

Therefore

$$\mathbb{P}(N_t \geq k) = (1-p_t)(1-q_t) \sum_{j=k}^{\infty} q_t^{j-1} = (1-p_t)q_t^{k-1} = \frac{(\beta-\alpha)e^{(\beta-\alpha)t}\beta^{k-1}(e^{(\beta-\alpha)t}-1)^{k-1}}{(\beta e^{(\beta-\alpha)t}-\alpha)^k}.$$

Also, since $\mathbb{P}[N_t^{(k)}] = \lim_{\theta \uparrow 1} \frac{\partial^k F(\theta, t)}{\partial \theta^k}$,

$$(14) \quad \mathbb{P}[N_t^{(k)}] = \frac{(1-p_t)(1-q_t)}{q_t} \frac{q_t^k k!}{(1-q_t)^{k+1}} = k! \left(\frac{\beta}{\beta-\alpha} \right)^{k-1} e^{(\beta-\alpha)t} (e^{(\beta-\alpha)t}-1)^{k-1}.$$

Thus

$$(15) \quad \frac{\mathbb{P}[N_t^{(k)}]}{\mathbb{P}(N_t \geq k)} = \frac{k!(\beta e^{(\beta-\alpha)t}-\alpha)^k}{(\beta-\alpha)^k} \quad \text{and} \quad \frac{\mathbb{P}[N_{T-t}^{(k)}]}{\mathbb{P}[N_T^{(k)}]} = e^{-(\beta-\alpha)t} \left(\frac{e^{(\beta-\alpha)(T-t)}-1}{e^{(\beta-\alpha)T}-1} \right)^{k-1}.$$

Finally we see that

$$(16) \quad \frac{\partial F(\theta, t)}{\partial t} = - \frac{(\beta-\alpha)^2(\beta\theta-\alpha)(1-\theta)e^{(\beta-\alpha)t}}{(\beta(1-\theta)e^{(\beta-\alpha)t}+\beta\theta-\alpha)^2}.$$

In the critical case $\alpha = \beta$, similar calculations give

$$(17) \quad F(\theta, t) = \frac{(1-\theta)\beta t + \theta}{(1-\theta)\beta t + 1}, \quad \mathbb{P}[N_t^{(k)}] = k!(\beta t)^{k-1}, \quad \frac{\mathbb{P}[N_t^{(k)}]}{\mathbb{P}(N_t \geq k)} = k!(\beta t + 1)^k$$

and

$$(18) \quad \frac{\partial F(\theta, t)}{\partial t} = \frac{\partial}{\partial t} \left(1 + \frac{\theta-1}{(1-\theta)\beta t + 1} \right) = \frac{(1-\theta)^2\beta}{((1-\theta)\beta t + 1)^2}.$$

5.2. Split time densities. Recall that \mathcal{H}' is the σ -algebra that contains information about which marks follow which spines, but does not know anything about the spine split times.

LEMMA 18. *Under $\mathbb{Q}^{k,T}$, the spine split times $\psi_1, \dots, \psi_{k-1}$ are independent of \mathcal{H}' and have a joint probability density function*

$$f_k^{\mathbb{Q}}(s_1, \dots, s_{k-1}) = \begin{cases} (k-1)! \left(\frac{\beta-\alpha}{e^{(\beta-\alpha)T}-1} \right)^{k-1} \prod_{i=1}^{k-1} e^{(\beta-\alpha)(T-s_i)} & \text{if } \alpha \neq \beta \\ (k-1)!/T^{k-1} & \text{if } \alpha = \beta \end{cases}.$$

PROOF. We do the calculation in the non-critical case $\alpha \neq \beta$. The proof in the critical case is identical. Recall from Lemma 9 that

$$\mathbb{Q}^{k,T}(\psi_1 > s_1) = \frac{\mathbb{P}[N_{T-s_1}^{(k)}]}{\mathbb{P}[N_T^{(k)}]} e^{(m-1)rs_1} = \frac{\mathbb{P}[N_{T-s_1}^{(k)}]}{\mathbb{P}[N_T^{(k)}]} e^{(\beta-\alpha)s_1}.$$

Then the second part of (15) gives

$$\mathbb{Q}^{k,T}(\psi_1 > s_1) = e^{-(\beta-\alpha)s_1} \left(\frac{e^{(\beta-\alpha)(T-s_1)}-1}{e^{(\beta-\alpha)T}-1} \right)^{k-1} e^{(\beta-\alpha)s_1} = \left(\frac{e^{(\beta-\alpha)(T-s_1)}-1}{e^{(\beta-\alpha)T}-1} \right)^{k-1},$$

so ψ_1 has density

$$(k-1)(\beta-\alpha)e^{(\beta-\alpha)(T-s_1)} \frac{(e^{(\beta-\alpha)(T-s_1)} - 1)^{k-2}}{(e^{(\beta-\alpha)T} - 1)^{k-1}}.$$

For $i = 2, \dots, k-1$, between times ψ_{i-1} and ψ_i we have exactly i particles carrying marks. Let A_i be the event that the first of these is carrying a_1 marks, the second a_2 , and so on. Let $\psi_i^{(j)}$ be the time at which the marks following the j th of these particles split. By the symmetry lemma, given $\psi_{i-1} = s_{i-1}$ (where we take $s_0 = 0$), these times are independent with

$$\mathbb{Q}^{k,T}(\psi_i^{(j)} > s_i | \psi_{i-1} = s_{i-1}, A_i) = \mathbb{Q}^{a_j, T-s_{i-1}}(\psi_1 > s_i - s_{i-1}) = \left(\frac{e^{(\beta-\alpha)(T-s_i)} - 1}{e^{(\beta-\alpha)(T-s_{i-1})} - 1} \right)^{a_j-1}.$$

Then, since the event $\{\psi_i > s_i\} = \bigcap_j \{\psi_i^{(j)} > s_i\}$,

$$\mathbb{Q}^{k,T}(\psi_i > s_i | \psi_{i-1} = s_{i-1}, A_i) = \prod_{j=1}^i \left(\frac{e^{(\beta-\alpha)(T-s_i)} - 1}{e^{(\beta-\alpha)(T-s_{i-1})} - 1} \right)^{a_j-1}.$$

Since $\sum_{j=1}^i (a_j - 1) = k - i$, we get

$$\mathbb{Q}^{k,T}(\psi_i > s_i | \psi_{i-1} = s_{i-1}, A_i) = \left(\frac{e^{(\beta-\alpha)(T-s_i)} - 1}{e^{(\beta-\alpha)(T-s_{i-1})} - 1} \right)^{k-i}.$$

This does not depend on a_1, \dots, a_i , so ψ_i is independent of \mathcal{H}' , and summing over the possible values we obtain

$$\mathbb{Q}^{k,T}(\psi_i > s_i | \psi_{i-1} = s_{i-1}) = \left(\frac{e^{(\beta-\alpha)(T-s_i)} - 1}{e^{(\beta-\alpha)(T-s_{i-1})} - 1} \right)^{k-i}.$$

Differentiating gives

$$f_k^{\mathbb{Q}}(s_1, \dots, s_{k-1}) = (k-1)!(\beta-\alpha)^{k-1} \prod_{i=1}^{k-1} e^{(\beta-\alpha)(T-s_i)} \frac{(e^{(\beta-\alpha)(T-s_i)} - 1)^{k-i-1}}{(e^{(\beta-\alpha)(T-s_{i-1})} - 1)^{k-i}}.$$

The product telescopes to give the answer. □

PROPOSITION 19. *Let $s_0 = 0$. The vector $(\mathcal{S}_1^k(T), \dots, \mathcal{S}_{k-1}^k(T))$ of ordered split times under \mathbb{P} is independent of \mathcal{H} and has a joint density $f_k^T(s_1, \dots, s_{k-1})$ equalling*

$$\begin{cases} \frac{k!(\beta e^{(\beta-\alpha)T} - \alpha)^k (\beta - \alpha)^{2k-1}}{(e^{(\beta-\alpha)T} - 1)^{k-1} e^{(\beta-\alpha)T}} \int_0^1 (1-y)^{k-1} \prod_{j=0}^{k-1} \frac{e^{(\beta-\alpha)(T-s_j)}}{(\beta(1-y)e^{(\beta-\alpha)(T-s_j)} + \beta y - \alpha)^2} dy, & \alpha \neq \beta \\ \frac{k!(\beta T + 1)^k}{T^{k-1}} \int_0^1 (1-y)^{k-1} \prod_{j=0}^{k-1} \frac{1}{(\beta(1-y)(T-s_j) + 1)^2} dy, & \alpha = \beta. \end{cases}$$

PROOF. Again we give the proof in the non-critical case $\alpha \neq \beta$. The critical case is identical. We start with Proposition 7, which tells us that for any measurable functional F ,

$$(19) \quad \mathbb{P}\left[\frac{1}{N_T^{(k)}} \sum_{u \in \mathcal{N}_T^{(k)}} F(u) \mid N_T \geq k\right] = \frac{\mathbb{P}[N_T^{(k)}]}{\mathbb{P}(N_T \geq k)(k-1)!} \int_0^\infty (e^z - 1)^{k-1} \mathbb{Q}^{k,T}\left[e^{-zN_T} F(\xi_T)\right] dz.$$

The independence of the spine split times and \mathcal{H}' under $\mathbb{Q}^{k,T}$ (established in Lemma 18), together with (19) and Proposition 16, imply that the split times under \mathbb{P} are independent of \mathcal{H} . (As a reminder, $\mathcal{H} = \sigma(P_1^k(T), \dots, P_{k-1}^k(T))$, so that \mathcal{H} contains all the topological information about the tree generated by U_T^1, \dots, U_T^k , but almost no information about the split times.)

Returning to (19) again and applying it with F equal to the indicator that the split times of the k -tuple are in (ds_1, \dots, ds_{k-1}) , we find that $f_k^T(s_1, \dots, s_{k-1})$ is equal to

$$\frac{\mathbb{P}[N_T^{(k)}]}{\mathbb{P}(N_T \geq k)(k-1)!} \int_0^\infty (e^z - 1)^{k-1} f_k^{\mathbb{Q}}(s_1, \dots, s_{k-1}) \mathbb{Q}[e^{-zN_T} | \psi_1 = s_1, \dots, \psi_{k-1} = s_{k-1}] dz.$$

However we also know from Proposition 16 that

$$\mathbb{Q}[e^{-z\tilde{N}_T} | \psi_1 = s_1, \dots, \psi_{k-1} = s_{k-1}] = \prod_{i=0}^{k-1} \left(e^{-r(m-1)(T-s_i)} \frac{u(F(e^{-z}, T-s_i))}{u(e^{-z})} \right)$$

where $s_0 = 0$, $F(\theta, t) = \mathbb{P}[\theta^{N_t}]$ and $u(\theta) = \mathbb{P}[\theta^L] - \theta$. Of course since all births are binary, all particles are either spines or ordinary; so since there are k spines at time T almost surely under \mathbb{Q} , $N_T = \tilde{N}_T + k$. Thus, by (13) and (16),

$$\mathbb{Q}[e^{-zN_T} | \psi_1 = s_1, \dots, \psi_{k-1} = s_{k-1}] = e^{-zk} \prod_{i=0}^{k-1} \left(\frac{\beta - \alpha}{\beta(1 - e^{-z})e^{(\beta-\alpha)(T-s_i)} + \beta e^{-z} - \alpha} \right)^2.$$

Plugging this into our formula for $f_k^T(s_1, \dots, s_{k-1})$ above gives

$$\begin{aligned} f_k^T(s_1, \dots, s_{k-1}) &= \frac{\mathbb{P}[N_T^{(k)}]}{\mathbb{P}(N_T \geq k)(k-1)!} \int_0^\infty e^{-z} (1 - e^{-z})^{k-1} f_k^{\mathbb{Q}}(s_1, \dots, s_{k-1}) \\ &\quad \cdot \prod_{i=0}^{k-1} \frac{(\beta - \alpha)^2}{(\beta(1 - e^{-z})e^{(\beta-\alpha)(T-s_i)} + \beta e^{-z} - \alpha)^2} dz. \end{aligned}$$

By the first part of (15) and Lemma 18, this becomes

$$\frac{k!(\beta e^{(\beta-\alpha)T} - \alpha)^k (\beta - \alpha)^{2k-1}}{e^{(\beta-\alpha)T} (e^{(\beta-\alpha)T} - 1)^{k-1}} \int_0^\infty e^{-z} (1 - e^{-z})^{k-1} \prod_{i=0}^{k-1} \frac{e^{(\beta-\alpha)(T-s_i)}}{(\beta(1 - e^{-z})e^{(\beta-\alpha)(T-s_i)} + \beta e^{-z} - \alpha)^2} dy.$$

Making the substitution $y = e^{-z}$ gives the result. \square

5.3. *Describing the partition process and proving Theorems 1 and 2.* We recall now the partition Z_0, Z_1, \dots which contained the information about the marks following each of the distinct spine particles, without the information about the split times.

LEMMA 20. *The partition Z_0, Z_1, \dots has the following distribution under $\mathbb{Q}_T^{k,T}$:*

- If Z_i consists of $i + 1$ blocks of sizes a_1, \dots, a_{i+1} , then the j th block will split next with probability $(a_j - 1)/(k - i - 1)$ for each $j = 1, \dots, i + 1$.
- When a block of size a splits, it splits into two new blocks, and the probability that these blocks have sizes l and $a - l$ is $1/(a - 1)$ for each $l = 1, \dots, a - 1$.

PROOF. Suppose that we are given $\psi_i = s$. For the first part, by the symmetry lemma, the probability that the j th block splits next is

$$\int_0^{T-s} \mathbb{Q}^{a_j, T-s}(\psi_1 \in dt) \prod_{l \neq j} \mathbb{Q}^{a_l, T-s}(\psi_l > t)$$

which by Lemma 9 equals

$$\int_0^{T-s} \left(-\frac{d}{dt} \left(\frac{\mathbb{P}[N_{T-s-t}^{(a_j)}]}{\mathbb{P}[N_{T-s}^{(a_j)}]} e^{(m-1)rt} \right) \right) \prod_{l \neq j} \frac{\mathbb{P}[N_{T-s-t}^{(a_l)}]}{\mathbb{P}[N_{T-s}^{(a_l)}]} e^{(m-1)rt} dt.$$

If $\alpha \neq \beta$, then applying the second part of (15), the above becomes

$$\begin{aligned} & \int_0^{T-s} \left(-\frac{d}{dt} \left(\frac{e^{(\beta-\alpha)(T-s-t)} - 1}{e^{(\beta-\alpha)(T-s)} - 1} \right)^{a_j-1} \right) \prod_{l \neq j} \left(\frac{e^{(\beta-\alpha)(T-s-t)} - 1}{e^{(\beta-\alpha)(T-s)} - 1} \right)^{a_l-1} dt \\ &= (a_j - 1)(\beta - \alpha) \int_0^{T-s} e^{(\beta-\alpha)(T-s-t)} \frac{(e^{(\beta-\alpha)(T-s-t)} - 1)^{a_j-2}}{(e^{(\beta-\alpha)(T-s)} - 1)^{a_j-1}} \prod_{l \neq j} \left(\frac{e^{(\beta-\alpha)(T-s-t)} - 1}{e^{(\beta-\alpha)(T-s)} - 1} \right)^{a_l-1} dt \\ &= (a_j - 1)(\beta - \alpha) \int_0^{T-s} \frac{e^{(\beta-\alpha)(T-s-t)}}{e^{(\beta-\alpha)(T-s-t)} - 1} \left(\frac{e^{(\beta-\alpha)(T-s-t)} - 1}{e^{(\beta-\alpha)(T-s)} - 1} \right)^{k-i-1} dt. \end{aligned}$$

Since the integrand does not depend on a_j , and we know the sum of the above quantity over $j = 1, \dots, i+1$ must equal 1 (since one of the blocks must split first), we get

$$(\beta - \alpha) \int_0^{T-s} \frac{e^{(\beta-\alpha)(T-s-t)}}{e^{(\beta-\alpha)(T-s-t)} - 1} \left(\frac{e^{(\beta-\alpha)(T-s-t)} - 1}{e^{(\beta-\alpha)(T-s)} - 1} \right)^{k-i-1} dt = \frac{1}{k-i-1}$$

and therefore the probability that the j th block splits next equals $\frac{a_j-1}{k-i-1}$ as claimed. If $\alpha = \beta$ then applying (17) in place of (15) gives the same result.

For the second part, let ρ_t^1 be the number of marks following the first spine particle at time t . From the definition of $\mathbb{Q}^{k,T}$,

$$\mathbb{Q}^{k,T}(\rho_t^1 = i \mid \tau_\emptyset = t) = \frac{\mathbb{P}[g_{k,T} \mathbb{1}_{\{\rho_t^1 = i\}} \mid \tau_\emptyset = t]}{\mathbb{P}[g_{k,T} \mid \tau_\emptyset = t]}.$$

By the Markov property, since each mark chooses uniformly from amongst the children available,

$$\mathbb{P}[g_{k,T} \mathbb{1}_{\{\rho_t^1 = i\}} \mid \tau_\emptyset = t] = \frac{\beta}{\beta + \alpha} \binom{k}{i} \mathbb{P}[g_{i,T-t}] \mathbb{P}[g_{k-i,T-t}].$$

Lemma 6 tells us that $\mathbb{P}[g_{j,s}] = \mathbb{P}[N_s^{(j)}]$ for any j and s , so

$$\mathbb{P}[g_{k,T} \mathbb{1}_{\{\rho_t^1 = i\}} \mid \tau_\emptyset = t] = \frac{\beta}{\beta + \alpha} \binom{k}{i} \mathbb{P}[N_{T-t}^{(i)}] \mathbb{P}[N_{T-t}^{(k-i)}].$$

If $\alpha \neq \beta$, then applying (14) gives

$$\begin{aligned} \mathbb{P}[g_{k,T} \mathbb{1}_{\{\rho_t^1 = i\}} \mid \tau_\emptyset = t] &= \frac{\beta}{\beta + \alpha} \binom{k}{i} i! (k-i)! \left(\frac{\beta}{\beta - \alpha} \right)^{k-2} e^{(\beta-\alpha)(T-t)} (e^{(\beta-\alpha)(T-t)} - 1)^{k-2} \\ &= \frac{\beta}{\beta + \alpha} k! \left(\frac{\beta}{\beta - \alpha} \right)^{k-2} e^{(\beta-\alpha)(T-t)} (e^{(\beta-\alpha)(T-t)} - 1)^{k-2}. \end{aligned}$$

Since this does not depend upon i , the distribution of ρ_t^1 under $\mathbb{Q}^{k,T}$ must be uniform. The case $\alpha = \beta$ uses (17) in place of (14). The result now follows from the symmetry lemma. \square

PROOF OF THEOREM 1. By Proposition 19, the *ordered* split times are independent of \mathcal{H} and have density (conditional on $N_T \geq k$)

$$\frac{k!(\beta E_0 - \alpha)^k(\beta - \alpha)^{2k-1}}{(E_0 - 1)^{k-1}E_0} \int_0^1 (1-y)^{k-1} \prod_{j=0}^{k-1} \frac{e^{(\beta-\alpha)(T-s_j)}}{(\beta(1-y)e^{(\beta-\alpha)(T-s_j)} + \beta y - \alpha)^2} dy$$

for any $0 \leq s_1 \leq \dots \leq s_{k-1} \leq 1$, where $s_0 = 0$. Therefore (see [11, Lemma 36] for details) the *unordered* split times are independent of \mathcal{H} and have density $1/(k-1)!$ times the above. Integrating over s_j for each $j = 1, \dots, k-1$ (see [11, Lemma 35] for details), we get

$$\begin{aligned} & \mathbb{P}(\tilde{\mathcal{S}}_1 \geq s_1, \dots, \tilde{\mathcal{S}}_{k-1} \geq s_{k-1} | N_T \geq k) \\ &= \frac{k(\beta E_0 - \alpha)^k(\beta - \alpha)}{(E_0 - 1)^{k-1}E_0} \int_0^1 (1-y)^{k-1} \left(\prod_{j=1}^{k-1} \frac{E_j - 1}{\beta(1-y)E_j + \beta y - \alpha} \right) \frac{E_0}{(\beta(1-y)E_0 + \beta y - \alpha)^2} dy. \end{aligned}$$

Substituting $\theta = 1 - y$, it is an elementary task to calculate this integral and deduce the desired result. For the details we refer to [11]. \square

The proof of the critical case, Theorem 2, is almost identical. It is written in full in [11].

6. The near-critical scaling limit. We now let our offspring distribution depend on T , writing \mathbb{P}_T in place of \mathbb{P} . We suppose that Assumption 1 holds, i.e. that $m_T := \mathbb{P}_T[L] = 1 + \mu/T + o(1/T)$ for some $\mu \in \mathbb{R}$, $\mathbb{P}_T[L(L-1)] = \sigma^2 + o(1)$ for some $\sigma > 0$, and L^2 is uniformly integrable (that is, for all $\varepsilon > 0$ there exists M such that $\sup_T \mathbb{P}_T[L^2 \mathbb{1}_{\{L \geq M\}}] < \varepsilon$). We define $\mathbb{Q}_T^{k,T}$ just as before, except that it is defined relative to \mathbb{P}_T^k instead of \mathbb{P}^k .

In order to prove our results we would like some conditions on the higher moments of L . Besides Assumption 1 we will further assume that there exists a deterministic sequence $J(T) = o(T)$ such that $\mathbb{P}_T(L = j) = 0$ for all $j \geq J(T)$. In particular this implies that for any $j \geq 3$,

$$(20) \quad \mathbb{P}_T[L^{(j)}] = \sum_{i=1}^{J(T)} i^{(j)} p_i^{(T)} \leq J(T)^{j-2} \sum_{i=1}^{J(T)} i(i-1) p_i^{(T)} = J(T)^{j-2} (\sigma^2 + o(1)) = o(T^{j-2}).$$

In fact there is no loss of generality in making this further assumption: by [11, Lemma 22], there exists a coupling between any tree satisfying Assumption 1 and a tree also satisfying the assumption above, such that for each k , conditionally on $N_T \geq k$, the two trees are equal until time T with probability tending to 1 as $T \rightarrow \infty$.

6.1. *Estimating moments and generating functions under \mathbb{P} .* The exact calculations carried out in Section 5.1 are no longer possible with our more complicated offspring distributions. Instead the near-criticality ensures that we can give good approximations.

LEMMA 21. *For $k \geq 1$, the k th descending moment $M_k(t) = \mathbb{P}[N_t^{(k)}]$ of any continuous-time Galton-Watson process satisfies*

$$M'_k(t) = kr(m-1)M_k(t) + r \sum_{j=2}^k \binom{k}{j} \mathbb{P}[L^{(j)}] M_{k+1-j}(t).$$

PROOF. This is an elementary application of the Kolmogorov forward equation (13). We omit it here; the details are available in [11, Lemma 23]. \square

LEMMA 22. *The descending moments at scaled times satisfy, for all $k \geq 1$ and $s \in [0, 1]$,*

$$\lim_{T \rightarrow \infty} \frac{\mathbb{P}_T[N_{sT}^{(k)}]}{T^{k-1}} = \begin{cases} \left(\frac{\sigma^2}{2\mu}\right)^{k-1} k! e^{r\mu s} (e^{r\mu s} - 1)^{k-1} & \text{if } \mu \neq 0 \\ k! \left(\frac{r\sigma^2 s}{2}\right)^{k-1} & \text{if } \mu = 0. \end{cases}$$

PROOF. We proceed by induction. Note that both statements are true for $k = 1$. Letting $M_k(t) = \mathbb{P}_T[N_t^{(k)}]$, by Lemma 21 we have

$$M'_k(t) = kr(m_T - 1)M_k(t) + r \sum_{j=2}^k \binom{k}{j} \mathbb{P}_T[L^{(j)}] M_{k+1-j}(t).$$

Letting $\hat{M}_k(s) = M_k(sT)$ and using the induction hypothesis, we have

$$\begin{aligned} \hat{M}'_k(s) &= T \left(kr(m_T - 1)\hat{M}_k(s) + r \sum_{j=2}^k \binom{k}{j} \mathbb{P}_T[L^{(j)}] \hat{M}_{k+1-j}(s) \right) \\ (21) \quad &= kr\mu \hat{M}_k(s) + Tr \binom{k}{2} \sigma^2 \hat{M}_{k-1}(s) + o(T^{k-1}). \end{aligned}$$

We now consider the cases $\mu \neq 0$ and $\mu = 0$ separately. In the case $\mu \neq 0$, using the integrating factor $e^{-kr\mu s}$, and applying the induction hypothesis again, we get

$$(22) \quad \frac{d}{ds} (e^{-kr\mu s} \hat{M}_k(s)) = T^{k-1} k! (k-1) r \mu \left(\frac{\sigma^2}{2\mu}\right)^{k-1} e^{-(k-1)r\mu s} (e^{r\mu s} - 1)^{k-2} + e^{-kr\mu s} O(T^{k-2}).$$

Noting that $(k-1)r\mu e^{-(k-1)r\mu s} (e^{r\mu s} - 1)^{k-2} = \frac{d}{ds} (e^{-(k-1)r\mu s} (e^{r\mu s} - 1)^{k-1})$, by integrating (22) we obtain

$$e^{-kr\mu s} \hat{M}_k(s) = T^{k-1} k! \left(\frac{\sigma^2}{2\mu}\right)^{k-1} e^{-(k-1)r\mu s} (e^{r\mu s} - 1)^{k-1} + e^{-kr\mu s} O(T^{k-2}).$$

Multiplying through by $e^{kr\mu s}$ gives the result for $\mu \neq 0$. If $\mu = 0$, then from (21) and the induction hypothesis, we have

$$\hat{M}'_k(s) = T^{k-1} k! \left(\frac{r\sigma^2}{2}\right)^{k-1} (k-1) s^{k-2} + o(T^{k-1})$$

and integrating directly gives the result. □

6.2. *Asymptotics for the generating function.* Define

$$F_T(\theta, t) = \mathbb{P}_T[\theta^{N_t}], \quad u_T(\theta) = \mathbb{P}_T[\theta^L] - \theta,$$

$$\text{and } f_T(\phi, s) = T(1 - \mathbb{P}_T[e^{-\frac{\phi}{T} N_{sT}}]) = T(1 - F_T(e^{-\phi/T}, sT)).$$

LEMMA 23. *For each $\phi \geq 0$, in the limit as $T \rightarrow \infty$,*

$$f_T(\phi, s) \rightarrow f(\phi, s) \quad \text{and} \quad T^2 u_T(F_T(e^{-\phi/T}, sT)) \rightarrow -\mu f(\phi, s) + \frac{\sigma^2}{2} f(\phi, s)^2$$

uniformly over $s \in [0, 1]$, where

$$f(\phi, s) = \frac{\phi e^{\mu r s}}{1 + \frac{\sigma^2}{2\mu} \phi (e^{\mu r s} - 1)} \quad \text{if } \mu \neq 0, \quad \text{and} \quad f(\phi, s) = \frac{\phi}{1 + r\sigma^2 \phi s/2} \quad \text{if } \mu = 0.$$

PROOF. First we show that for each ϕ , f_T is bounded in $T > 0$ and $s \in [0, 1]$. Note that $x \mapsto 1 - e^{-\kappa x}$ is concave and increasing for any $\kappa \geq 0$, so by Jensen's inequality,

$$f_T(\phi, s) = T(1 - \mathbb{P}[e^{-\frac{\phi}{T} N_{sT}}]) \leq T(1 - e^{-\frac{\phi}{T} \mathbb{P}_T[N_{sT}]}) \leq T(1 - e^{-\frac{\phi}{T} \exp(r\mu + o(1))}).$$

Applying the inequality $1 - e^{-x} \leq x$, we see that $f_T(\phi, s) \leq \phi e^{r\mu + o(1)}$. Now, with $F_T(\theta, t) = \mathbb{P}_T[\theta^{N_t}]$, we have

$$(23) \quad \frac{\partial f_T(\phi, s)}{\partial s} = \frac{\partial}{\partial s}(T(1 - F_T(e^{-\phi/T}, sT))) = -T^2 \frac{\partial F_T(e^{-\phi/T}, t)}{\partial t} \Big|_{t=sT}.$$

By the Kolmogorov backwards equation (13),

$$(24) \quad \frac{\partial}{\partial t} F_T(\theta, t) = ru_T(F_T(\theta, t)) = r\mathbb{P}_T[F_T(\theta, t)^L] - rF_T(\theta, t),$$

so

$$\frac{\partial f_T(\phi, s)}{\partial s} = T^2 r \sum_{j=0}^{\infty} p_j^{(T)} (F(e^{-\phi/T}, sT) - F(e^{-\phi/T}, sT)^j) = T^2 r \sum_{j=0}^{\infty} p_j^{(T)} \left(1 - \frac{f_T}{T} - \left(1 - \frac{f_T}{T}\right)^j\right)$$

where $p_j^{(T)} = \mathbb{P}_T(L = j)$. Expanding $(1 - f_T/T)^j$, we get

$$\begin{aligned} \frac{\partial f_T(\phi, s)}{\partial s} &= T^2 r \sum_{j=0}^{\infty} p_j^{(T)} \left((j-1) \frac{f_T}{T} - \frac{j(j-1)f_T^2}{2T^2} - \sum_{i=3}^j \binom{j}{i} \left(-\frac{f_T}{T}\right)^i \right) \\ &= r\mu f_T - \frac{r\sigma^2}{2} f_T^2 + o(1) - T^2 r \sum_{j=0}^{\infty} p_j^{(T)} \sum_{i=3}^j \binom{j}{i} \left(-\frac{f_T}{T}\right)^i. \end{aligned}$$

Swapping the order of summation, this becomes

$$\begin{aligned} \frac{\partial f_T(\phi, s)}{\partial s} &= r\mu f_T - \frac{r\sigma^2}{2} f_T^2 + o(1) - T^2 r \sum_{i=3}^{\infty} \frac{1}{i!} \left(-\frac{f_T}{T}\right)^i \sum_{j=i}^{\infty} p_j^{(T)} j(j-1) \dots (j-i+1) \\ (25) \quad &= r\mu f_T - \frac{r\sigma^2}{2} f_T^2 + o(1) \end{aligned}$$

since f_T is bounded and $\mathbb{P}_T[L^{(i)}] = o(T^{i-2})$ for each $i \geq 3$ by (20). Note in particular that the $o(1)$ term is uniform in s .

Note that f is the solution to

$$\frac{\partial f}{\partial s} = r\mu f - \frac{r\sigma^2}{2} f^2$$

with $f(\phi, 0) = \phi$. Setting $h_T(\phi, s) = f_T(\phi, s) - f(\phi, s)$ we have

$$\frac{\partial h_T}{\partial s} = r\mu(f_T - f) - \frac{r\sigma^2}{2}(f_T^2 - f^2) + o(1)$$

where the $o(1)$ term is uniform in s . Integrating over s with ϕ fixed,

$$h_T(\phi, s) = h_T(\phi, 0) + r\mu \int_0^s h_T(\phi, s') ds' - \frac{r\sigma^2}{2} \int_0^s h_T(\phi, s')(f_T(\phi, s') + f(\phi, s')) ds + o(1).$$

For fixed ϕ , both f_T and f are bounded in s and T , say by M_ϕ . Also $|h_T(\phi, 0)| = T(1 - e^{-\phi/T}) - \phi = o(1)$. Thus

$$|h_T(\phi, s)| \leq r \int_0^s |h_T(\phi, s')|(\mu + \sigma^2 M_\phi/2) ds' + o(1),$$

where again the $o(1)$ term is uniform in s . Gronwall's inequality then tells us that $|h_T(\phi, s)| \rightarrow 0$ uniformly in s . This proves the first part of the lemma.

The second part is now implicit in our calculations above: by (24) and then (23),

$$u_T(F_T(e^{-\phi/T}, sT)) = \frac{1}{r} \frac{\partial}{\partial t} F_T(e^{-\phi/T}, t)|_{t=sT} = -\frac{1}{rT^2} \frac{\partial f_T(\phi, s)}{\partial s}.$$

Applying (25) tells us that

$$T^2 u_T(F_T(e^{-\phi/T}, sT)) = -\mu f_T + \frac{\sigma^2}{2} f_T^2 + o(1),$$

and by the first part of the lemma we get

$$T^2 u_T(F_T(e^{-\phi/T}, sT)) \rightarrow -\mu f + \frac{\sigma^2}{2} f^2. \quad \square$$

Our next lemma is not new. The critical case goes back to Kolmogorov [15] (under a third moment condition, which has since been relaxed by other authors) and the non-critical case is [22, Theorem 2.1(i)]. For the non-critical case we give a self-contained proof below which does not rely on the diffusion approximation used in [22].

LEMMA 24. *For any $s \in (0, 1]$, as $T \rightarrow \infty$,*

$$T\mathbb{P}_T(N_{sT} > 0) \rightarrow \frac{2\mu e^{\mu rs}}{\sigma^2(e^{\mu rs} - 1)} \quad \text{if } \mu \neq 0, \quad \text{and} \quad T\mathbb{P}_T(N_{sT} > 0) \rightarrow \frac{2}{r\sigma^2 s} \quad \text{if } \mu = 0.$$

PROOF. Note that $\mathbb{P}_T(N_t = 0) = F_T(0, t)$, and so satisfies the Kolmogorov backwards equation (13). Thus the proof of Lemma 23 works exactly the same for

$$T\mathbb{P}_T(N_{sT} > 0) = T(1 - \mathbb{P}_T(N_{sT} = 0)) = T(1 - F_T(0, sT)),$$

except for showing $T\mathbb{P}_T(N_{sT} > 0)$ is bounded as we can no longer apply Jensen's inequality.

Instead, we note that in the critical case $m_T = 1$ the boundedness is well known (see for example [5, Chapter III, Section 7, Lemma 2]). When $m_T \neq 1$, let

$$\bar{p}_0^{(T)} = p_0^{(T)} - (1 - m_T) \log(3/2)$$

and for $j \geq 1$,

$$\bar{p}_j^{(T)} = p_j^{(T)} + (1 - m_T) 2^{-j}/j.$$

This gives us a new offspring distribution \bar{L} that is critical (and has finite variance). We can then easily construct a coupling between N_t and \bar{N}_t , where \bar{N}_t is the number of particles in a branching process with offspring distribution \bar{L} , such that

- if $m_T < 1$, then $N_t \leq \bar{N}_t$ for all $t \geq 0$;
- if $m_T > 1$, then $N_t \geq \bar{N}_t$ for all $t \geq 0$.

In the case $m_T < 1$, we have $T\mathbb{P}(N_{sT} > 0) \leq T\mathbb{P}(\bar{N}_{sT} > 0)$, which is bounded. In the case $m_T > 1$, we have

$$\mathbb{P}_T(N_{sT} > 0) = \mathbb{Q}_T^{1,sT} \left[\frac{\mathbb{P}_T[N_{sT}]}{N_{sT}} \right] = e^{r(m_T-1)sT} \mathbb{Q}_T^{1,sT} \left[\frac{1}{N_{sT}} \right]$$

and similarly for \bar{N}_{sT} with its equivalent measure $\bar{\mathbb{Q}}_T^{1,sT}$. Since $T\mathbb{P}(\bar{N}_{sT} > 0)$ is bounded, we get that $T\bar{\mathbb{Q}}_T^{1,sT}[1/\bar{N}_{sT}]$ is bounded, but

$$\mathbb{Q}_T^{1,sT} \left[\frac{1}{N_{sT}} \right] \leq \bar{\mathbb{Q}}_T^{1,sT} \left[\frac{1}{\bar{N}_{sT}} \right],$$

so $T\mathbb{Q}_T^{1,sT}[1/N_{sT}]$ is bounded and therefore $T\mathbb{P}_T(N_{sT} > 0)$ is also bounded. \square

6.3. Spine split times under $\mathbb{Q}_T^{k,T}$. We now want to feed our calculations for moments and generating functions under \mathbb{P} into understanding the spine split times under \mathbb{Q} , as in Lemma 18. Unfortunately the spine split times in non-binary cases do not have a joint density with respect to Lebesgue measure: for any $j = 2, \dots, k-1$, there is a positive probability that $\psi_j = \psi_{j-1}$. However we show that this probability tends to zero as $T \rightarrow \infty$, and therefore will not have an effect on our final answer.

Recall that n_t is the number of distinct spine particles at time t , and ρ_i^t is the number of marks carried by spine i at time t .

LEMMA 25. *For any $i = 1, \dots, k-1$ and $t \in (0, 1)$,*

$$\mathbb{Q}_T^{k,T} \left(n_{\psi_1} = 2, \rho_{\psi_1}^1 = i \mid \frac{\psi_1}{T} = t \right) \rightarrow \frac{1}{k-1}.$$

This tells us two things: that with probability tending to 1 we have exactly 2 spines at the first spine split time; and that the number of marks following each of those spines is uniformly distributed on $1, \dots, k-1$.

PROOF. We work in the case $\mu \neq 0$; the case $\mu = 0$ proceeds almost identically. From the definition of \mathbb{Q} ,

$$\mathbb{Q}_T^{k,T}(n_{tT} = 2, \rho_{tT}^1 = i \mid \tau_\emptyset = tT, n_{tT} \geq 2) = \frac{\mathbb{P}_T[g_{k,T} \mathbb{1}_{\{n_{tT}=2, \rho_{tT}^1=i\}} \mid \tau_\emptyset = tT]}{\mathbb{P}_T[g_{k,T} \mathbb{1}_{\{n_{tT} \geq 2\}} \mid \tau_\emptyset = tT]}.$$

Let $P_T(j; b; a_1, \dots, a_b)$ be the probability that at time τ_\emptyset , j children are born, b of which are spines, carrying a_1, \dots, a_b marks. Then

$$\mathbb{P}_T[g_{k,T} \mathbb{1}_{\{n_{tT}=b, \rho_{tT}^1=a_1\}} \mid \tau_\emptyset = tT] = \sum_{j=b}^{\infty} \sum_{a_2, \dots, a_b} P_T(j; b; a_1, \dots, a_b) j^k \prod_{i=1}^b \mathbb{P}_T[g_{a_i, T(1-t)}]$$

where the sum over a_2, \dots, a_b runs over $1, \dots, k$ such that $a_1 + \dots + a_b = k$. Now

$$P_T(j; b; a_1, \dots, a_b) = p_j^{(T)} \binom{j}{b} \frac{k!}{a_1! \dots a_b!} \frac{1}{j^k}$$

and from Lemma 22, in the case $\mu \neq 0$,

$$\mathbb{P}_T[N_{T(1-t)}^{(a_i)}] = T^{a_i-1} \left(\frac{\sigma^2}{2\mu} \right)^{a_i-1} a_i! e^{r\mu(1-t)} (e^{r\mu(1-t)} - 1)^{a_i-1} + o(T^{a_i-1}).$$

This gives us

$$\begin{aligned} \mathbb{P}_T[g_{k,T} \mathbb{1}_{\{n_{tT}=b, \rho_{tT}^1=a_1\}} | \tau_\emptyset = tT] \\ = \sum_{j=b}^{\infty} \sum_{a_2, \dots, a_b} p_j^{(T)} \binom{j}{b} k! T^{k-b} \left(\frac{\sigma^2}{2\mu}\right)^{k-b} e^{br\mu(1-t)} (e^{r\mu(1-t)} - 1)^{k-b} (1 + o(1)). \end{aligned}$$

If $b = 2$, then fixing $a_1 = i$ also fixes a_2 since $a_2 = k - a_1$, so the second sum disappears and we are left with

$$\begin{aligned} \mathbb{P}_T[g_{k,T} \mathbb{1}_{\{n_{tT}=2, \rho_{tT}^1=i\}} | \tau_\emptyset = tT] \\ = \sum_{j=2}^{\infty} p_j^{(T)} \binom{j}{2} k! T^{k-2} \left(\frac{\sigma^2}{2\mu}\right)^{k-2} e^{2r\mu(1-t)} (e^{r\mu(1-t)} - 1)^{k-2} (1 + o(1)) \\ (26) \quad = \frac{\sigma^2}{2} k! T^{k-2} \left(\frac{\sigma^2}{2\mu}\right)^{k-2} e^{2r\mu(1-t)} (e^{r\mu(1-t)} - 1)^{k-2} (1 + o(1)). \end{aligned}$$

Notice in particular that this does not depend on the value of i .

Next we bound the probability that there are at least three distinct spines at time ψ_1 by taking a sum over a_1 and then over $b \geq 3$. For each b , there are certainly at most k^b possible values of a_1, \dots, a_b that sum to k . Thus we get

$$\mathbb{P}_T[g_{k,T} \mathbb{1}_{\{n_{tT} \geq 3\}} | \tau_\emptyset = tT] \leq \sum_{b=3}^{\infty} \mathbb{P}_T[L^{(b)}] \frac{k!}{b!} k^b T^{k-b} \left(\frac{\sigma^2}{2\mu}\right)^{k-b} e^{br\mu(1-t)} (e^{r\mu(1-t)} - 1)^{k-b} (1 + o(1)).$$

Recall that we have assumed (20) that $\mathbb{P}_T[L^{(b)}] = o(T^{b-2})$ for each $b \geq 3$, so

$$(27) \quad \mathbb{P}_T[g_{k,T} \mathbb{1}_{\{n_{tT} \geq 3\}} | \tau_\emptyset = tT] = o(T^{k-2}).$$

Dividing (27) by (26), we see that the probability that there are at least 3 distinct spines at time ψ_1 tends to zero as $T \rightarrow \infty$; or equivalently, that the probability that there are exactly 2 distinct spines tends to 1. Then since the right-hand side of (26) does not depend on i , the distribution of ρ_{ψ_1} must be asymptotically uniform. \square

Combined with the symmetry lemma, the previous result tells us that with high probability the spine split times are distinct. We want to use this to show that away from 0, the rescaled split times $\psi_1/T, \dots, \psi_{k-1}/T$ have an asymptotic density. First we need a preparatory lemma, which will be helpful in describing the topology of our limiting tree as well as calculating the asymptotic density of the split times.

LEMMA 26. *For any $s \in (0, 1]$ and $t \in (0, s)$, as $T \rightarrow \infty$,*

$$\mathbb{Q}_T^{k,sT} \left(\frac{\psi_1}{T} > t \right) \rightarrow \left(\frac{e^{r\mu(s-t)} - 1}{e^{r\mu s} - 1} \right)^{k-1}$$

and

$$-\frac{d}{dt} \mathbb{Q}_T^{k,sT} \left(\frac{\psi_1}{T} > t \right) \rightarrow (k-1)r\mu \frac{(e^{r\mu(s-t)} - 1)^{k-2}}{(e^{r\mu s} - 1)^{k-1}} e^{r\mu(s-t)}.$$

PROOF. The first part follows easily by combining Lemmas 9 and 22. The second part needs more calculation. As in Lemma 21, we write $M_k(t) = \mathbb{P}_T[N_t^{(k)}]$. By Lemma 9,

$$\mathbb{Q}_T^{k,sT}(\psi_1 > tT) = \frac{\mathbb{P}[N_{T(s-t)}^{(k)}]}{\mathbb{P}[N_{sT}^{(k)}]} e^{(m_T-1)rtT} = \frac{M_k(T(s-t))}{M_k(sT)} e^{(m_T-1)rtT},$$

so

$$\begin{aligned} -\frac{d}{dt} \mathbb{Q}_T^{k,sT}(\psi_1 > tT) &= T \frac{M'_k(T(s-t))}{M_k(sT)} e^{(m_T-1)rtT} - T(m_T-1)r \frac{M_k(T(s-t))}{M_k(sT)} e^{(m_T-1)rtT} \\ &= \frac{T}{M_k(sT)} e^{(m_T-1)rtT} (M'_k(T(s-t)) - (m_T-1)rM_k(T(s-t))). \end{aligned}$$

Applying Lemma 21, this equals

$$\frac{T}{M_k(sT)} e^{(m_T-1)rtT} \left((k-1)r(m_T-1)M_k(T(s-t)) + r \sum_{j=2}^k \binom{k}{j} \mathbb{P}_T[L^{(j)}] M_{k+1-j}(T(s-t)) \right).$$

We now use Lemma 22. Since $\mathbb{P}_T[L^{(j)}] = o(T^{j-2})$ for all $j \geq 3$ (see (20)), the terms with $j \geq 3$ in the sum above do not contribute in the limit. We obtain

$$\begin{aligned} \frac{T e^{r\mu t}}{(\frac{\sigma^2}{2\mu})^{k-1} k! e^{r\mu s} (e^{r\mu s} - 1)^{k-1} T^{k-1}} &\left[(k-1)r\mu \left(\frac{\sigma^2}{2\mu}\right)^{k-1} k! e^{r\mu(s-t)} (e^{r\mu(s-t)} - 1)^{k-1} T^{k-2} \right. \\ &\left. + r \frac{k(k-1)}{2} \sigma^2 \left(\frac{\sigma^2}{2\mu}\right)^{k-2} (k-1)! e^{r\mu(s-t)} (e^{r\mu(s-t)} - 1)^{k-2} T^{k-2} + o(T^{k-2}) \right]. \end{aligned}$$

Simplifying, this equals

$$\frac{1}{(e^{r\mu s} - 1)^{k-1}} \left[(k-1)r\mu (e^{r\mu(s-t)} - 1)^{k-1} + (k-1)r\mu (e^{r\mu(s-t)} - 1)^{k-2} + o(1) \right],$$

so simplifying again we get

$$-\frac{d}{dt} \mathbb{Q}_T^{k,sT}(\psi_1 > tT) \rightarrow (k-1)r\mu \frac{(e^{r\mu(s-t)} - 1)^{k-2}}{(e^{r\mu s} - 1)^{k-1}} e^{r\mu(s-t)}. \quad \square$$

Recall that \mathcal{H}' is the σ -algebra containing topological information about which marks are following which spines, without information about the spine split times.

PROPOSITION 27. *The spine split times $\psi_1, \dots, \psi_{k-1}$ are asymptotically independent of \mathcal{H}' under $\mathbb{Q}_T^{k,T}$, and for any $0 < s_1 < t_1 \leq s_2 < t_2 \leq \dots \leq s_{k-1} < t_{k-1} < 1$,*

$$\lim_{T \rightarrow \infty} \mathbb{Q}_T^{k,T} \left(\frac{\psi_1}{T} \in (s_1, t_1], \dots, \frac{\psi_{k-1}}{T} \in (s_{k-1}, t_{k-1}] \right) = \int_{s_1}^{t_1} \dots \int_{s_{k-1}}^{t_{k-1}} f_k(s'_1, \dots, s'_{k-1}) ds'_{k-1} \dots ds'_1,$$

where

$$f_k(s_1, \dots, s_{k-1}) = \begin{cases} (k-1)! \left(\frac{r\mu}{e^{r\mu}-1} \right)^{k-1} \prod_{i=1}^{k-1} e^{r\mu(1-s_i)} & \text{if } \mu \neq 0 \\ (k-1)! & \text{if } \mu = 0. \end{cases}$$

PROOF. This is a generalization of the proof of Lemma 18, and the reader may wish to compare the two. The main difference is that now there is a chance that spine splitting events result in more than one new spine particle (since branching events need not be binary), and therefore we need to take care to ensure that the split times $\psi_1, \dots, \psi_{k-1}$ are distinct.

With this in mind, let Υ_j be the event that the first j spine split times are distinct,

$$\Upsilon_j = \{\psi_i \neq \psi_{i-1} \ \forall i = 2, \dots, j\}.$$

We work by induction; fix $j \leq k-1$, $T > 0$, $0 < s_1 < \dots < s_{j-1} < 1$. Then for $s \geq s_{j-1}$,

$$\begin{aligned} \mathbb{Q}\left(\frac{\psi_j}{T} > s \mid \frac{\psi_{j-1}}{T} = s_{j-1}, \dots, \frac{\psi_1}{T} = s_1\right) &= \mathbb{Q}\left(\Upsilon_j, \frac{\psi_j}{T} > s \mid \frac{\psi_{j-1}}{T} = s_{j-1}, \dots, \frac{\psi_1}{T} = s_1\right) \\ &= \mathbb{Q}\left(\frac{\psi_j}{T} > s \mid \Upsilon_j, \frac{\psi_{j-1}}{T} = s_{j-1}, \dots, \frac{\psi_1}{T} = s_1\right) \mathbb{Q}\left(\Upsilon_j \mid \frac{\psi_{j-1}}{T} = s_{j-1}, \dots, \frac{\psi_1}{T} = s_1\right). \end{aligned}$$

By Lemma 25 and the symmetry lemma,

$$\mathbb{Q}\left(\Upsilon_j \mid \frac{\psi_{j-1}}{T} = s_{j-1}, \dots, \frac{\psi_1}{T} = s_1\right) \rightarrow 1$$

for all $0 < s_1 < \dots < s_{j-1} < 1$. We also set

$$D(s) = -\frac{d}{ds} \mathbb{Q}\left(\frac{\psi_j}{T} > s \mid \Upsilon_j, \frac{\psi_{j-1}}{T} = s_{j-1}, \dots, \frac{\psi_1}{T} = s_1\right)$$

and claim that

$$D(s) = (k-j)r\mu e^{r\mu(1-s)} \frac{(e^{r\mu(1-s)} - 1)^{k-j-1}}{(e^{r\mu(1-s_{j-1})} - 1)^{k-j}} + o(1).$$

If this claim holds, then applying induction and taking a product over j gives the result. In particular, since this does not depend on the number of marks following each spine, the split times are asymptotically independent of \mathcal{H}' .

To prove the claim, fix a_1, \dots, a_j such that $a_i \in \{1, \dots, k\}$ for each i and $a_1 + \dots + a_j = k$. Let A_j be the event that after time ψ_{j-1} , we have j distinct spine particles carrying a_1, \dots, a_j marks. Then by the symmetry lemma (letting $s_0 = 0$),

$$\mathbb{Q}_T^{k,T}\left(\frac{\psi_j}{T} > s_j \mid \Upsilon_j, A_j, \frac{\psi_{j-1}}{T} = s_{j-1}\right) = \prod_{i=1}^j \mathbb{Q}_T^{a_i, T(1-s_{j-1})}(\psi_1/T > s_j - s_{j-1}).$$

Thus, differentiating, we have

$$D(s) = -\sum_{a_1, \dots, a_j} P_{a_1, \dots, a_j} \sum_{l=1}^j \left(\frac{d}{ds} \mathbb{Q}_T^{a_l, T(1-s_{j-1})}\left(\frac{\psi_l}{T} > s - s_{j-1}\right) \right) \prod_{i \neq l} \mathbb{Q}_T^{a_i, T(1-s_{j-1})}\left(\frac{\psi_i}{T} > s - s_{j-1}\right)$$

where P_{a_1, \dots, a_j} is the probability that A_j occurs. Applying Lemma 26 then establishes the claim and completes the proof. \square

We recall now the partition Z_0, Z_1, \dots which contained the information about the marks following each of the distinct spine particles, without the information about the split times.

LEMMA 28. *The partition Z_0, Z_1, \dots has the following distribution under $\mathbb{Q}_T^{k,T}$:*

- If Z_i consists of $i + 1$ blocks of sizes a_1, \dots, a_{i+1} , then the j th block will split next with probability $\frac{a_j-1}{k-i-1}(1 + o(1))$ for each $j = 1, \dots, i + 1$.
- When a block of size a splits, it splits into two new blocks with probability $1 + o(1)$, and the probability that these blocks have sizes l and $a - l$ is $\frac{1}{a-1}(1 + o(1))$ for each $l = 1, \dots, a - 1$.

PROOF. Suppose that we are given $\psi_i = sT$. For the first part, by the symmetry lemma, the probability that the j th block splits next is

$$\begin{aligned} \int_0^{T(1-s)} \mathbb{Q}_T^{a_j, T(1-s)}\left(\frac{\psi_1}{T} \in dt\right) \prod_{l \neq j} \mathbb{Q}_T^{a_l, T(1-s)}\left(\frac{\psi_1}{T} > t\right) \\ = \int_0^{T(1-s)} \left(-\frac{d}{dt} \mathbb{Q}_T^{a_j, T(1-s)}\left(\frac{\psi_1}{T} > t\right)\right) \prod_{l \neq j} \mathbb{Q}_T^{a_l, T(1-s)}\left(\frac{\psi_1}{T} > t\right) dt. \end{aligned}$$

By Lemma 26, this converges as $T \rightarrow \infty$ to

$$(a_j - 1)r\mu \int_0^{T(1-s)} e^{r\mu(1-s-t)} \frac{e^{(r\mu(1-s-t) - 1)^{k-i}}}{e^{(r\mu(1-s) - 1)^{k-i-1}}} dt.$$

Since the integrand does not depend on a_j , and we know the sum of the above quantity over $j = 1, \dots, i + 1$ must converge to 1 (since one of the blocks must split first), we get

$$r\mu \int_0^{T(1-s)} e^{r\mu(1-s-t)} \frac{e^{(r\mu(1-s-t) - 1)^{k-b-1}}}{e^{(r\mu(1-s) - 1)^{k-b}}} dt \rightarrow \frac{1}{k - i - 1}$$

and therefore the probability that the j th block splits next converges to $\frac{a_j-1}{k-i-1}$ as claimed.

The second part follows immediately from Lemma 25. \square

6.4. *Asymptotics for N_T under $\mathbb{Q}_T^{k,T}$.* We now apply our asymptotics for $u_T(F_T(e^{-z}, sT))$ to approximate the distribution of N_T when the split times are known.

LEMMA 29. For any $\phi \geq 0$ and $0 \leq s_1 \leq \dots \leq s_{k-1} \leq 1$,

$$\mathbb{Q}_T^{k,T}\left[e^{-\phi \tilde{N}_T/T} \mid \mathcal{G}_T^k, \frac{\psi_1}{T} = s_1, \dots, \frac{\psi_{k-1}}{T} = s_{k-1}\right] \rightarrow \begin{cases} \prod_{i=0}^{k-1} \left(1 + \frac{\sigma^2}{2\mu} \phi (e^{r\mu(1-s_i)} - 1)\right)^{-2} & \text{if } \mu \neq 0 \\ \prod_{i=0}^{k-1} \left(1 + \frac{r\sigma^2}{2} \phi (1 - s_i)\right)^{-2} & \text{if } \mu = 0 \end{cases}$$

almost surely as $T \rightarrow \infty$.

PROOF. From Proposition 16 we know that

$$\mathbb{Q}_T^{k,T}\left[e^{-\phi \tilde{N}_T/T} \mid \mathcal{G}_T^k, \frac{\psi_i}{T} = s_i, i \leq k-1\right] = \prod_{i=0}^{k-1} \left(e^{-r(m_T-1)T(1-s_i)} \frac{u_T(F_T(e^{-\phi/T}, T(1-s_i)))}{u_T(e^{-\phi/T})}\right).$$

Of course $(m_T - 1)T \rightarrow \mu$, and Lemma 23 tells us that

$$T^2 u_T(F_T(e^{-\phi/T}, T(1-s_i))) \rightarrow -\mu f(\phi, 1-s_i) + \frac{\sigma^2}{2} f(\phi, 1-s_i)^2$$

where

$$f(\phi, s) = \frac{\phi e^{\mu r s}}{1 + \frac{\sigma^2}{2\mu} \phi (e^{\mu r s} - 1)} \quad \text{if } \mu \neq 0 \quad \text{or} \quad f(\phi, s) = \frac{\phi}{1 + \frac{r\sigma^2}{2} \phi s} \quad \text{if } \mu = 0.$$

Noting that $u_T(e^{-\phi/T}) = u_T(F_T(e^{-\phi/T}, 0))$, we see that

$$e^{-r(m_T-1)T(1-s_i)} \frac{u_T(F_T(e^{-\phi/T}, T(1-s_i)))}{u_T(e^{-\phi/T})} \rightarrow e^{-r\mu(1-s_i)} \frac{-\mu f(\phi, 1-s_i) + \frac{\sigma^2}{2} f(\phi, 1-s_i)^2}{-\mu f(\phi, 0) + \frac{\sigma^2}{2} f(\phi, 0)^2}.$$

Now, in the case $\mu \neq 0$, we simply write out

$$\begin{aligned} -\mu f(\phi, 1-s_i) + \frac{\sigma^2}{2} f(\phi, 1-s_i)^2 &= \frac{-\mu \phi e^{r\mu(1-s_i)} (1 + \frac{\sigma^2}{2\mu} \phi (e^{\mu r(1-s_i)} - 1)) + \frac{\sigma^2}{2} \phi^2 e^{2r\mu(1-s_i)}}{(1 + \frac{\sigma^2}{2\mu} \phi (e^{\mu r(1-s_i)} - 1))^2} \\ &= \frac{-\mu \phi e^{r\mu(1-s_i)} + \frac{\sigma^2}{2} \phi^2 e^{r\mu(1-s_i)}}{(1 + \frac{\sigma^2}{2\mu} \phi (e^{\mu r(1-s_i)} - 1))^2}, \end{aligned}$$

so since $-\mu f(\phi, 0) + \frac{\sigma^2}{2} f(\phi, 0)^2 = -\mu \phi + \sigma^2 \phi^2 / 2$, we have

$$e^{-r\mu(1-s_i)} \frac{-\mu f(\phi, 1-s_i) + \frac{\sigma^2}{2} f(\phi, 1-s_i)^2}{-\mu f(\phi, 0) + \frac{\sigma^2}{2} f(\phi, 0)^2} = \left(1 + \frac{\sigma^2}{2\mu} \phi (e^{\mu r(1-s_i)} - 1)\right)^{-2}.$$

The result in the case $\mu = 0$ is very similar. □

LEMMA 30. *For any $\phi \geq 0$, $\mathbb{Q}_T^{k,T}$ -almost surely,*

$$\mathbb{Q}_T^{k,T}[e^{-\phi(N_T-k)/T} | \mathcal{G}_T^k] = \mathbb{Q}_T^{k,T}[e^{-\phi \tilde{N}_T/T} | \mathcal{G}_T^k] (1 + o(1)).$$

PROOF. Recall that \tilde{N}_T is the number of ordinary particles alive at time T , and there are (\mathbb{Q} -almost surely) k spines at time T . All other particles are residue particles. Given \mathcal{G}_T^k , the number of residue particles is independent of the number of ordinary particles; therefore it suffices to show that

$$\mathbb{Q}^{k,T}[e^{-\phi(N_T-k-\tilde{N}_T)/T} | \mathcal{G}_T^k] \rightarrow 1.$$

Recall that we assumed that there exists a deterministic function $J(T) = o(T)$ such that our offspring distribution satisfies $\mathbb{P}_T(L = j) = 0$ for all $j \geq J(T)$. Since $\mathbb{Q}^{k,T}$ is absolutely continuous with respect to \mathbb{P}_T , we also have $\mathbb{Q}^{k,T}(L = j) = 0$ for all $j \geq J(T)$.

Since non-spine particles behave exactly as under \mathbb{P}_T , the generating function in z for the number of descendants at time T of any one particle born at time ψ_i is $\mathbb{P}_T[e^{-zN_T-s}]|_{s=\psi_i}$. Therefore

$$\mathbb{Q}_T^{k,T}[e^{-\phi(N_T-k-\tilde{N}_T)/T} | \mathcal{G}_T^k] \geq \prod_{i=1}^{k-1} \mathbb{P}_T[e^{-\phi N_T-s/T}]^{J(T)} \Big|_{s=\psi_i}.$$

By Jensen's inequality, for any $t \in [0, T]$,

$$\mathbb{P}_T[e^{-\phi N_t/T}] \geq \exp(-\phi \mathbb{P}_T[N_t]/T) \geq \exp(-\phi e^{r(m_T-1)T}/T),$$

and thus

$$\mathbb{Q}_T^{k,T}[e^{-\phi(N_T-k-\tilde{N}_T)/T} | \mathcal{G}_T^k] \geq \mathbb{P}_T[\exp(-\phi e^{r(m_T-1)T} J(T)/T)]^{k-1}.$$

Since $J(T) = o(T)$, the right-hand side converges to 1 as $T \rightarrow \infty$, and trivially we have that $\mathbb{Q}_T^{k,T}[e^{-\phi(N_T-k-\tilde{N}_T)/T} | \mathcal{G}_T^k] \leq 1$, so we are done. □

Recall that Υ_{k-1} is the event that all the split times are distinct, and \mathcal{H}' is the σ -algebra that contains topological information about which marks follow which spines without information about the spine split times. Let $(\tilde{\psi}_1, \dots, \tilde{\psi}_{k-1})$ be a uniform random permutation of $(\psi_1, \dots, \psi_{k-1})$. We combine several of our results to prove the following.

LEMMA 31. *Fix $s_1, \dots, s_{k-1} \in (0, 1)$. Let*

$$f(\xi_T) = \mathbb{1}_{\{\tilde{\psi}_1/T > s_1, \dots, \tilde{\psi}_{k-1}/T > s_{k-1}, \Upsilon_{k-1}\} \cap H}$$

where $H \in \mathcal{H}'$. There exists a constant h such that $\mathbb{Q}_T^{k,T}(H) \rightarrow h$ as $T \rightarrow \infty$. For any $\phi \geq 0$, if $\mu \neq 0$ then

$$\lim_{T \rightarrow \infty} \mathbb{Q}_T^{k,T}[e^{-\phi(N_T-k)/T} f(\xi_T)] = \left(\frac{1}{e^{r\mu} - 1}\right)^{k-1} \frac{h}{(1 + \frac{\sigma^2}{2\mu}\phi(e^{r\mu} - 1))^2} \prod_{i=1}^{k-1} \frac{e^{r\mu(1-s_i)} - 1}{1 + \frac{\sigma^2}{2\mu}\phi(e^{r\mu(1-s_i)} - 1)}$$

and if $\mu = 0$ then

$$\lim_{T \rightarrow \infty} \mathbb{Q}_T^{k,T}[e^{-\phi(N_T-k)/T} f(\xi_T)] = \frac{h}{(1 + r\sigma^2\phi/2)^2} \prod_{i=1}^{k-1} \frac{1 - s_i}{1 + r\sigma^2\phi(1 - s_i)/2}.$$

PROOF. The fact that $\mathbb{Q}_T^{k,T}(H)$ converges follows from Lemma 28. Now, by Proposition 27 (see also [11, Lemma 36]), in the case $\mu \neq 0$,

$$\begin{aligned} \mathbb{Q}_T^{k,T}[e^{-\phi(N_T-k)/T} f(\xi_T)] &= (1 + o(1)) \int_{s_1}^1 \cdots \int_{s_{k-1}}^1 \left(\frac{r\mu}{e^{r\mu} - 1}\right)^{k-1} \left(\prod_{i=1}^{k-1} e^{r\mu(1-s'_i)}\right) \\ &\quad \cdot \mathbb{Q}_T^{k,T}\left[\mathbb{1}_H \mathbb{Q}_T^{k,T}\left[e^{-\phi(N_T-k)/T} \mid \mathcal{G}_T^k, \frac{\tilde{\psi}_1}{T} = s'_1, \dots, \frac{\tilde{\psi}_1}{T} = s'_{k-1}\right]\right]. \end{aligned}$$

By Lemma 30, we may replace $N_T - k$ with \tilde{N}_T ; and then by Lemma 29, the above equals

$$\begin{aligned} (1 + o(1)) \int_{s_1}^1 \cdots \int_{s_{k-1}}^1 \left(\frac{r\mu}{e^{r\mu} - 1}\right)^{k-1} \left(\prod_{i=1}^{k-1} e^{r\mu(1-s'_i)}\right) \\ \cdot \mathbb{Q}_T^{k,T}(H) \prod_{j=0}^{k-1} \left(1 + \frac{\sigma^2}{2\mu}\phi(e^{r\mu(1-s'_j)} - 1)\right)^{-2} ds'_{k-1} \cdots ds'_1 \end{aligned}$$

almost surely. After some small rearrangements this becomes

$$(1 + o(1)) \left(\frac{r\mu}{e^{r\mu} - 1}\right)^{k-1} \frac{h}{(1 + \frac{\sigma^2}{2\mu}\phi(e^{r\mu} - 1))^2} \prod_{i=1}^{k-1} \int_{s_i}^1 \frac{e^{r\mu(1-s'_i)}}{(1 + \frac{\sigma^2}{2\mu}\phi(e^{r\mu(1-s'_i)} - 1))^2} ds'_i,$$

and then integrating out over s'_i for each i gives the result (see [11, Lemma 35] for details). The case $\mu = 0$ is similar. \square

6.5. *The final steps in the proof of Theorem 3.*

PROOF OF THEOREM 3. Let

$$P_T(f, k) = \mathbb{P}_T \left[\frac{1}{N_T^{(k)}} \sum_{u \in \mathcal{N}_T^{(k)}} f(u) \mid N_T \geq k \right].$$

By Proposition 7, for any measurable f ,

$$P_T(f, k) = \frac{\mathbb{P}_T[N_T^{(k)}]}{\mathbb{P}_T(N_T \geq k)(k-1)!} \int_0^\infty (e^z - 1)^{k-1} \mathbb{Q}_T^{k,T} \left[e^{-zN_T} f(\xi_T) \right] dz.$$

Substituting $z = \phi/T$ and rearranging, this equals

$$\frac{1}{(k-1)!} \frac{\mathbb{P}_T[N_T^{(k)}]}{T^{k-1}} \frac{1}{T\mathbb{P}_T(N_T \geq k)} \int_0^\infty (T(1 - e^{-\phi/T}))^{k-1} \mathbb{Q}_T^{k,T} [e^{-\phi(N_T-k)/T} f(\xi_T)] e^{-\phi/T} d\phi.$$

By Lemma 22,

$$\frac{\mathbb{P}_T[N_T^{(k)}]}{T^{k-1}} \rightarrow \left(\frac{\sigma^2}{2\mu} \right)^{k-1} k! e^{r\mu} (e^{r\mu} - 1)^{k-1} \quad \text{if } \mu \neq 0 \quad \text{and} \quad \frac{\mathbb{P}_T[N_T^{(k)}]}{T^{k-1}} \rightarrow \left(\frac{r\sigma^2}{2} \right)^{k-1} k! \quad \text{if } \mu = 0,$$

and by Lemma 24,

$$T\mathbb{P}_T(N_T \geq k) \rightarrow \frac{2\mu e^{r\mu}}{\sigma^2(e^{r\mu} - 1)} \quad \text{if } \mu \neq 0 \quad \text{and} \quad T\mathbb{P}_T(N_T \geq k) \rightarrow \frac{2}{r\sigma^2} \quad \text{if } \mu = 0.$$

Therefore

$$\frac{1}{(k-1)!} \frac{\mathbb{P}_T[N_T^{(k)}]}{T^{k-1}} \frac{1}{T\mathbb{P}_T(N_T \geq k)} \rightarrow \begin{cases} k(\sigma^2/2\mu)^k (e^{r\mu} - 1)^k & \text{if } \mu \neq 0 \\ k(r\sigma^2/2)^k & \text{if } \mu = 0. \end{cases}$$

We deduce that, when $\mu \neq 0$,

$$(28) \quad P_T(f, k) = (1 + o(1)) k \left(\frac{\sigma^2}{2\mu} \right)^k (e^{r\mu} - 1)^k \int_0^\infty (T(1 - e^{-\phi/T}))^{k-1} \mathbb{Q}_T^{k,T} [e^{-\phi(N_T-k)/T} f(\xi_T)] e^{-\phi/T} d\phi$$

and, when $\mu = 0$,

$$P_T(f, k) = (1 + o(1)) k \left(\frac{r\sigma^2}{2} \right)^k \int_0^\infty (T(1 - e^{-\phi/T}))^{k-1} \mathbb{Q}_T^{k,T} [e^{-\phi(N_T-k)/T} f(\xi_T)] e^{-\phi/T} d\phi.$$

Our aim now is to choose f as in Lemma 31, and apply dominated convergence and Lemma 31 to complete the proof. We do this only in the case $\mu \neq 0$; the case $\mu = 0$ is very similar. Let

$$A(\phi, T) = (T(1 - e^{-\phi/T}))^{k-1} \mathbb{Q}_T^{k,T} [e^{-\phi(N_T-k)/T} f(\xi_T)]$$

and

$$B(\phi, T) = (T(1 - e^{-\phi/T}))^{k-1} \mathbb{Q}_T^{k,T} [e^{-\phi(N_T-k)/T}].$$

Then $0 \leq A(\phi, T) \leq B(\phi, T)$ for all ϕ, T . By letting $s_1, \dots, s_{k-1} \downarrow 0$ in Lemma 31, we have

$$\begin{aligned} \lim_{T \rightarrow \infty} \mathbb{Q}_T^{k,T} [e^{-\phi(N_T-k)/T} \mathbb{1}_{\mathcal{R}_{k-1}}] &= \left(\frac{1}{e^{r\mu} - 1} \right)^{k-1} \frac{1}{(1 + \frac{\sigma^2}{2\mu} \phi (e^{r\mu} - 1))^2} \left(\frac{e^{r\mu} - 1}{1 + \frac{\sigma^2}{2\mu} \phi (e^{r\mu} - 1)} \right)^{k-1} \\ &= \left(1 + \frac{\sigma^2}{2\mu} \phi (e^{r\mu} - 1) \right)^{-(k+1)}. \end{aligned}$$

Also, by Lemma 25, $\mathbb{Q}_T^{k,T}[e^{-\phi(N_T-k)/T} \mathbb{1}_{\Upsilon_{k-1}^c}] \leq \mathbb{Q}_T^{k,T}(\Upsilon_{k-1}^c) \rightarrow 0$, so

$$\lim_{T \rightarrow \infty} B(\phi, T) = \phi^{k-1} \left(1 + \frac{\sigma^2}{2\mu} \phi(e^{r\mu} - 1)\right)^{-(k+1)}.$$

On the other hand, by (28) with $f \equiv 1$,

$$1 = P_T(1, k) = (1 + o(1))k \left(\frac{\sigma^2}{2\mu}\right)^k (e^{r\mu} - 1)^k \int_0^\infty B(\phi, T) d\phi,$$

so

$$\lim_{T \rightarrow \infty} \int_0^\infty B(\phi, T) d\phi = \frac{1}{k} \left(\frac{2\mu}{\sigma^2(e^{r\mu} - 1)}\right)^k;$$

and as a result we see that $\lim_{T \rightarrow \infty} \int_0^\infty B(\phi, T) d\phi = \int_0^\infty \lim_{T \rightarrow \infty} B(\phi, T) d\phi$. Therefore, by dominated convergence,

$$(29) \quad \lim_{T \rightarrow \infty} \int_0^\infty A(\phi, T) d\phi = \int_0^\infty \lim_{T \rightarrow \infty} A(\phi, T) d\phi.$$

Lemma 31 tells us that

$$A(\phi, T) \rightarrow \left(\frac{\phi}{e^{r\mu} - 1}\right)^{k-1} \frac{h}{(1 + \frac{\sigma^2}{2\mu} \phi(e^{r\mu} - 1))^2} \prod_{i=1}^{k-1} \frac{e^{r\mu(1-s_i)} - 1}{1 + \frac{\sigma^2}{2\mu} \phi(e^{r\mu(1-s_i)} - 1)}$$

where $h = \lim_{T \rightarrow \infty} \mathbb{Q}_T^{k,T}(H)$, so by (28) and (29), as $T \rightarrow \infty$

$$\begin{aligned} P_T(f, k) &\rightarrow k \left(\frac{\sigma^2}{2\mu}\right)^k (e^{r\mu} - 1) \int_0^\infty \phi^{k-1} \frac{h}{(1 + \frac{\sigma^2}{2\mu} \phi(e^{r\mu} - 1))^2} \prod_{i=1}^{k-1} \frac{e^{r\mu(1-s_i)} - 1}{1 + \frac{\sigma^2}{2\mu} \phi(e^{r\mu(1-s_i)} - 1)} d\phi \\ &= \frac{k\sigma^2}{2\mu} (e^{r\mu} - 1) \int_0^\infty \frac{h}{(1 + \frac{\sigma^2}{2\mu} \phi(e^{r\mu} - 1))^2} \prod_{i=1}^{k-1} \left(1 - \frac{1}{1 + \frac{\sigma^2}{2\mu} \phi(e^{r\mu(1-s_i)} - 1)}\right) d\phi. \end{aligned}$$

Note that, for any $\mu \neq 0$, we have $\frac{\sigma^2}{2\mu} (e^{r\mu(1-s_i)} - 1) > 0$ for all i ; elementary calculations (see [11, Lemma 34] for details) give

$$\lim_{T \rightarrow \infty} P_T(f, k) = hk \left(\prod_{i=1}^{k-1} \frac{e_i}{e_i - e_0} \right) + hke_0 \sum_{j=1}^{k-1} \frac{e_j}{(e_j - e_0)^2} \left(\prod_{\substack{i=1 \\ i \neq j}}^{k-1} \frac{e_i}{e_i - e_j} \right) \log \frac{e_0}{e_j}$$

where $e_j = \frac{\sigma^2}{2\mu} (e^{r\mu(1-s_j)} - 1)$ for each j (including $j = 0$, where $s_0 = 0$). \square

6.6. Proof of construction of the scaling limit. In this section we prove Theorems 4 and 5.

PROOF OF THEOREM 4. Relabel $T_1, \dots, T_{I-1}, T_{I+1}, \dots, T_k$ as $\tilde{T}_1, \dots, \tilde{T}_{k-1}$. Since $\mathbb{P}(M_k \leq \theta) = \mathbb{P}(X_1 \leq \theta)^k$, we have

$$\begin{aligned} \mathbb{P}(\tilde{T}_1 \in ds_1, \dots, \tilde{T}_{k-1} \in ds_{k-1}) &= \int_0^\infty \mathbb{P}(M_k \in d\theta) \mathbb{P}(\tilde{T}_1 \in ds_1, \dots, \tilde{T}_{k-1} \in ds_{k-1} | M_k = \theta) \\ &= \int_0^\infty k \mathbb{P}(X_1 \in d\theta) \mathbb{P}(X_1 \leq \theta)^{k-1} \mathbb{P}(1 - \frac{X_1}{\theta} \in ds_1, \dots, 1 - \frac{X_{k-1}}{\theta} \in ds_{k-1} | X_1 \leq \theta, \dots, X_{k-1} \leq \theta) \\ &= \int_0^\infty \frac{k}{(1 + \theta)^2} \mathbb{P}(X_1 \leq \theta)^{k-1} \prod_{i=1}^{k-1} \mathbb{P}\left(1 - \frac{X_i}{\theta} \in ds_i | X_i \leq \theta\right) d\theta \\ &= \int_0^\infty \frac{k}{(1 + \theta)^2} \prod_{i=1}^{k-1} \mathbb{P}\left(1 - \frac{X_i}{\theta} \in ds_i\right) d\theta = \int_0^\infty \frac{k}{(1 + \theta)^2} \left(\prod_{i=1}^{k-1} \frac{\theta}{(1 + \theta(1 - s_i))^2} ds_i \right) d\theta. \end{aligned}$$

This is exactly the density that we saw for $(\tilde{\mathcal{S}}_1^k, \dots, \tilde{\mathcal{S}}_{k-1}^k)$ in Section 2.3.

To see that our tree has the topology claimed, start by assigning k marks to the top of the tallest line, i.e. at the point $(U_I, 1 - T_I)$. Colour this line green. Next colour the second tallest line blue; let its index be J . Since it is positioned uniformly on the horizontal axis, the number L of shorter lines to its left is uniformly distributed on $\{0, \dots, k - 2\}$, and so is the number $k - 2 - L$ to its right. Suppose without loss of generality that the blue line is to the left of the green line, and assign $L + 1$ marks to the top of the blue line, i.e. at point $(U_J, 1 - T_J)$, and $k - (L + 1)$ marks to the point $(U_I, 1 - T_J)$. (If the blue line were to the right of the green line, we would assign $k - (L + 1)$ marks to $(U_J, 1 - T_J)$ and $L + 1$ marks to $(U_I, 1 - T_J)$.) Thus the number of marks assigned to the top of the blue line is uniform on $\{1, \dots, k - 1\}$.

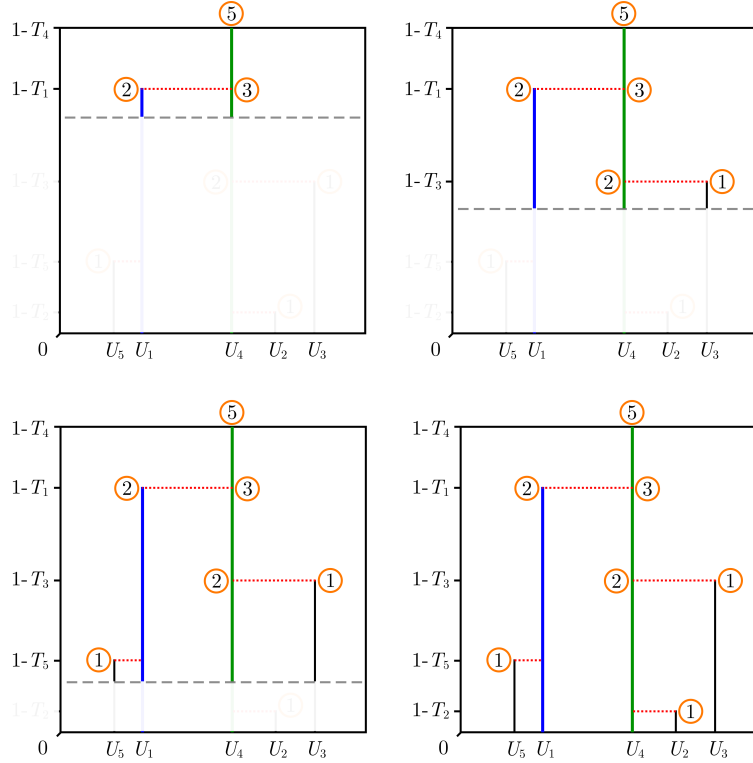


Fig 5: Constructing the tree by moving downwards through our picture. The number of marks are shown in circles. The as yet “unseen” parts of the tree are left blank. Here $k = 5$, $I = 4$ and $J = 1$.

Moving downwards, the next horizontal line to appear corresponds to the third-tallest vertical line. We ask which of the two coloured lines this next horizontal line will join to, i.e. which of the branches in the tree will split next. By our construction, the event that the third tallest line joins the blue line (given that the blue line is left of the green line) is exactly the event that the third tallest line is left of the blue line. Since the lengths of the branches are independent and identically distributed, this has probability $L/(k - 2)$. Furthermore, observe that the position of the third tallest line, conditionally on it falling to the left (respectively right) of the blue line, is uniformly distributed on $(0, U_J)$ (respectively $(U_J, 1)$).

More generally, once we have seen the n tallest vertical lines, and assigned a_i marks to line i for each line i that we have seen, the $(n + 1)$ st tallest vertical line has probability $(a_i - 1)/(k - n)$ of joining line i ; and the number of marks this new line gets is uniformly distributed on $\{1, \dots, a_i - 1\}$. This is exactly the topology outlined in Theorem 3. \square

PROOF OF THEOREM 5. Rather than doing the calculation directly, this follows from Theorem 4 by noting that making the substitution

$$t_i = \frac{e^{r\mu} - e^{r\mu(1-s_i)}}{e^{r\mu} - 1}$$

in the density f_k recovers the critical case from the non-critical. \square

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